ISOLATED RUPTURE DEGREE OF TREES AND GEAR GRAPHS

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Abstract: The isolated rupture degree for a connected graph G is defined as $ir(G) = \max\{i(G-S)-|S|-m(G-S) : S \in C(G)\}$, where $i(G-S)$ and $m(G-S)$, respectively, denote the number of components which are isolated vertices and the order of a largest component in $G-S$. $C(G)$ denotes the set of all cut-sets of G. The isolated rupture degree is a new graph parameter which can be used to measure the vulnerability of networks. In this paper, we firstly give a recursive algorithm for computing the isolated rupture degree of trees, and determine the maximum and minimum isolated rupture degree of trees with given order and maximum degree. Then, the exact value of isolated rupture degree of gear graphs are given. In the final, we determine the rupture degree of the Cartesian product of two special graphs and a special permutation graph.

Key words: *Isolated rupture degree, vulnerability, ir-set, recursive algorithm, Cartesian product, gear graph*

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1. Introduction

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with vertex set V and edge set E. The number of vertices $|V|$ is known as the order of G. For $v \in V$, we denote the degree of v by $d_G(v)$. The maximum degree of a graph G is denoted by $\Delta(G)$. A vertex set $S \subseteq V(G)$ is a cut set of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. $C(G)$ denotes the set of all cut-sets of G. For $S \subseteq V(G)$, let $\omega(G-S)$, $i(G-S)$ and $m(G-S)$, respectively, denote the number of components, the number of components which are isolated vertices and the order of a largest component in $G-S$. We shall use $\lfloor x \rfloor$ for the largest integer less than or equal to a real number x. A Δ-*edge* is an edge which joins two vertices of degree Δ. A *leaf* is a vertex of degree 1. An edge incident with a leaf is called a *leaf-edge*. An edge is said to be *subdivided* when it is replaced by a path of length two connecting its ends, and the internal vertex in this path is a new vertex. A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set S is called a maximum independent set

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if G has no independent set S' with $|S'| > |S|$. The *independence number* of G, $\alpha(G)$, is the number of vertices in a maximum independent set of G. By $\kappa(G)$ we denote the connectivity of G. For any $V' \subseteq V$, we let $G[V']$ denote the induced subgraph of G. We use Bondy and Murty [2] for terminology and notations not defined here.

In an analysis of the vulnerability of networks to disruption, three important quantities (there may be others) are (1) the number of elements that are not functioning; (2) the number of remaining connected sub-networks; (3) the size of a largest remaining group within which mutual communication can still occur.

The communication network can be represented as an undirected and unweighted graph, where a processor (station) is represented as a vertex and a communication link between processors (stations) as an edge between corresponding vertices. If we use a graph to model a network, based on the above three quantities, a number of graph parameters, such as connectivity [2], toughness [5], scattering number [7, 17], integrity [1], tenacity [6], rupture degree [8, 10, 11, 12, 14] and their edge-analogues, have been proposed for measuring the vulnerability of networks.

One of the vulnerability parameters determined above is the scattering number that takes the quantities (1) and (2) into account. The scattering number was introduced by Jung in 1978 [7], and the scattering number of an incomplete connected graph G is defined as

$$s(G) = \max\{\omega(G - S) - |S| : S \in C(G), \omega(G - S) > 1\}.$$

Motivated from Jung's scattering number by replacing $\omega(G-S)$ with $i(G-S)$ in the above definition, Wang et.al [15] introduced the isolated scattering number, $is(G)$, as a new parameter to measure the vulnerability of a network. The isolated scattering number of an incomplete graph G is defined as

$$is(G) = \max\{i(G - S) - |S| : S \in C(G)\}.$$

The rupture degree is a measure which deals with all the quantities, (1), (2) and (3). The rupture degree of an incomplete graph is defined as

$$r(G) = \max\{\omega(G - S) - |S| - m(G - S) : S \in C(G), \omega(G - S) > 1\}.$$

Motivated from the concept of the isolated scattering number, it is natural for us to replace $\omega(G-S)$ with $i(G-S)$ in the above definition, we call this parameter the *isolated rupture degree* of graphs. Formally, the isolated rupture degree of an incomplete connected graph G is defined as follows.

Definition 1.1 [9] Let G be an incomplete graph. Then the *isolated rupture degree* $ir(G)$ of G is defined as

$$ir(G) = \max\{i(G - S) - |S| - m(G - S) : S \in C(G)\}.$$

Here, $i(G-S)$ and $m(G-S)$, respectively, denote the number of components which are isolated vertices and the order of a largest component in $G - S$. $C(G)$ denotes the set of all cut-sets of G.

In particular, the isolated rupture degree of a complete graph K_n is defined to be $1 - n$.

Definition 1.2 [9] Let G be an incomplete connected graph. A set $S \in C(G)$ is called an ir-set if it satisfies $ir(G) = i(G-S) - |S| - m(G-S)$.

Definition 1.3 [6] The *tenacity* of an incomplete connected graph G is defined as

$$T(G) = \min\left\{\frac{|S| + m(G-S)}{\omega(G-S)} : S \in C(G), \omega(G-S) > 1\right\}.$$

Here, $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$.

As a new graph parameter to measure the vulnerability of networks, isolated rupture degree, isolated scattering number, rupture degree and tenacity differ in showing the vulnerability of networks. This can be shown as follows. Consider the graphs G_1 and G_2 in Fig. 1, it is not difficult to check that $T(G_1) = T(G_2) = \frac{1}{2}$, but $ir(G_1) = 0 \neq 4 = ir(G_2)$. On the other hand, we consider graphs $G_3 = K_1 + (K_{n-b-1} \cup E_b)$ and $G_4 = K_2 + (K_{n-b-3} \cup E_{b+1})$. It is obvious that $is(G_3) = is(G_4) = b-1$, but $ir(G_3) = 2b-n+1 \neq 2b-n+3 = ir(G_4)$, where $b(b \leq n-4)$ is an positive integer. In the next, we consider graphs $G_5 = K_1 + (bK_2 \cup E_{n-2b-1})$ and $G_6 = K_1 + ((b-3)K_2 \cup K_3 \cup E_{n-2b+2})$. It is obvious that $r(G_5) = r(G_6) = n-b-4$, but $ir(G_5) = n-2b-3 \neq n-2b-1 = ir(G_6)$, where $b(4 \leq b \leq \lfloor \frac{n-2}{2} \rfloor)$ is a positive integer.

Hence, isolated rupture degree is a reasonable parameter for distinguishing the vulnerability of these graphs. It is easy to see that the less the isolated rupture degree of a network the more stable it is considered to be. In [7], the author gives formulas for the isolated rupture degree of join graphs and some bounds of the rupture degree. And the author also determines the isolated rupture degree of grids, and that of the hypercubes as a special case.

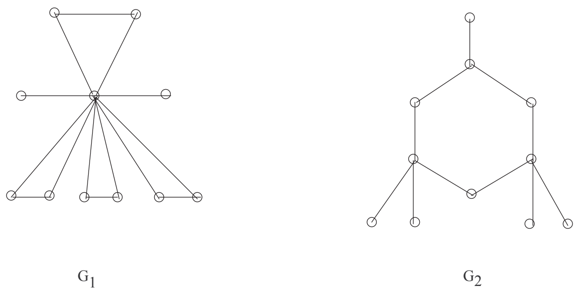

Fig. 1 *Two graphs G_1 and G_2 satisfy that $T(G_1) = T(G_2)$, but $ir(G_1) \neq ir(G_2)$.*

In this paper, we firstly give a recursive algorithm for computing the isolated rupture degree of trees, and determine the maximum and minimum isolated rupture degree of trees with given order and maximum degree. Then, the exact value of isolated rupture degree of gear graphs are given. In the final, we determine the rupture degree of the Cartesian product of two special graphs and a special permutation graph.

2. A recursive algorithm for computing the isolated rupture degree of trees

In computer science, tree is a commonly used data formation that feigns a hierarchical tree structure with a set of linked nodes. So, it is interesting for us to study the isolated rupture degree of trees. In this section, we provide a recursive algorithm for computing the isolated rupture degree of trees.

For a graph G, the *neighborhood* of vertex v, $N_G(v)$, is the set of vertices adjacent to v in G. A vertex $v \in V(T)$ is called an *outside branch vertex* of a tree T if it satisfies that $d_T(v) \geq 2$ and it is adjacent to $d_T(v) - 1$ number of leaves. For an outside branch vertex $v \in V(T)$, we denote $\Gamma^+(v) = \{u : u \in N_T(v), d_T(v) = 1\}$. It is obvious that $|\Gamma^+(v)| \geq 1$, and if the tree T is not a star, then $|V(T)| \geq 4$.

Lemma 2.1 [9] *Let $K_{1,n-1}$ and P_n be a star and a path of order $n \geq 3$, respectively. Then*

$$ir(K_{1,n-1}) = n - 3; \tag{1}$$

$$ir(P_n) = \begin{cases} -1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}, \tag{2}$$

Lemma 2.2 [9] *Let G be a connected bipartite graph. Then $ir(G) \geq -1$.*

Lemma 2.3 [9] *Let G be a connected bipartite graph of order n. Then $ir(G) = 2\alpha(G) - n - 1$.*

We have the following computational complexity result.

Theorem 2.1 [9] *Let G be a connected bipartite graph of order n with bipartition $[X, Y]$. Then the isolated rupture degree of G can be computed in $O(min\{|X|, |Y|\} \cdot |E(G)|)$ time.*

Lemma 2.4 *Let T be a tree of order $n(\geq 4)$. Then T has exactly one outside branch vertex if and only if $T = K_{1,n-1}$.*

Proof. The sufficiency is obvious by the definition of outside branch vertex. In the following, we will prove the necessity. We assume that $T \neq K_{1,n-1}$. Let S be the set of all leaves in T, and let $T' = T[V(T) - S]$. It is easy to see that T' is a tree such that $|V(T')| \geq 2$. Thus, T' have at least two leaves. Without loss of generality, we suppose that u and v be two distinct leaves in T'. By the definition, we know that u and v be two outside branch vertices in T, a contradiction. Thus, $T = K_{1,n-1}$. This completes the proof. □

By Lemma 2.4, we know that if $T \neq K_{1,n-1}$ be a tree of order $n \geq 4$, then T has at least two outside branch vertices. So, for any outside branch vertex $v \in V(T)$, $T_v = T - (\Gamma^+(v) \cup \{v\})$ is a tree such that $|V(T_v)| \geq 2$.

Theorem 2.2 *Let $T \neq K_{1,n-1}$ be a tree of order $n(\geq 4)$. Then for any outside branch vertex $v \in V(T)$,*

$$ir(T) = ir(T - (\Gamma^+(v) \cup \{v\})) + |\Gamma^+(v)| - 1.$$

Proof. Let $T_v = T - (\Gamma^+(v) \cup \{v\})$. By Lemma 2.4, T_v is a tree. Let S be a maximum independent set of T. Then, by Lemma 2.3, $ir(T) = 2|S| - n - 1$. And, it is easily seen that $S^* = S - (\Gamma^+(v) \cup \{v\})$ is an independent set of T_v. In the next, we will prove that $|S^*| = |S| - |\Gamma^+(v)|$. If $v \notin S$, then $\Gamma^+(v) \subseteq S$. This implies that $|S^*| = |S| - |\Gamma^+(v)|$. If $v \in S$, then $\Gamma^+(v) \cap S = \emptyset$. It is easy to see that $S' = (S \cup \Gamma^+(v)) - \{v\}$ is an independent set of T. Because S is a maximum independent set of T, then $|S| \geq |S'| = |S| + |\Gamma^+(v)| - |\{v\}| = |S| + |\Gamma^+(v)| - 1$, i.e., $|\Gamma^+(v)| \leq 1$. By the definition of outside branch vertex, we know that $|\Gamma^+(v)| \geq 1$. Hence, we know that $|\Gamma^+(v)| = 1$. Thus, in this case we have $|S^*| = |S| - 1 = |S| - |\Gamma^+(v)|$. So, it is proved that $|S^*| = |S| - |\Gamma^+(v)|$ whether $v \in S$ or not. Let $S'' = V(T_v) - S^*$. Then $|S''| = |V(T_v)| - |S^*| = |V(T)| - |S| - 1$, $i(T_v - S'') = |S^*|$, and $m(T_v - S'') = 1$. So, by the definition of isolated rupture degree and Lemma 2.3, we have

$$ir(T_v) \geq i(T_v - S'') - |S''| - 1 = 2|S| - |\Gamma^+(v)| - n = ir(T) - |\Gamma^+(v)| + 1.$$

On the other hand, let S' be a maximum independent set of T_v, then $S'' = S' \cup \Gamma^+(v)$ is an independent set of T. Let $S^* = V(T) - S''$. We know that $|S^*| = |V(T) - S''| = |V(T_v)| - |S'| + 1$, $i(T - S^*) = |S''| = |S'| + |\Gamma^+(v)|$, and $m(T - S^*) = 1$. By the definition of isolated rupture degree and Lemma 2.3, we get that

$$\begin{aligned} ir(T) &\geq i(T - S^*) - |S^*| - m(T - S^*) = 2|S'| - |V(T_v)| + |\Gamma^+(v)| - 2 \\ &= ir(T_v) + |\Gamma^+(v)| - 1. \end{aligned}$$

By combining the above two inequalities, we have

$$ir(T) = ir(T_v) + |\Gamma^+(v)| - 1.$$

This completes the proof. □

Note that any tree is a bipartite graph. By Theorem 2.1, we know that the isolated rupture degree of trees can be computed in polynomial time. Now, based on Lemma 2.1 and Theorem 2.2, we present a polynomial time recursive algorithm for computing the isolated rupture degree of trees. Firstly, let us describe the basic idea of the algorithm. If the tree is a path or a star, then its isolated rupture degree can be obtained by Lemma 2.1. Otherwise, by choosing an outside branch vertex vertex $v_1 \in V(T)$, we obtain a new tree $T_v = T - (\Gamma^+(v) \cup \{v\})$. If T_v is a star, then the isolated rupture degree of T_v is obtained. By Theorem 2.2, we can get the isolated rupture degree of T. Otherwise, select an outside branch vertex vertex $u \in V(T_{v_1})$. Then, we get a new tree $T_u = T_v - (\Gamma^+(u) \cup \{u\})$. Repeating the above process, we finally get a tree which is a star or $|V(T)| \leq 4$. Then the isolated rupture degree of T is obtained.

ALGORITHM 2.1

Input: A tree T such that $|V(T)| \geq 2$.

Output: Isolated rupture degree of tree T, $ir(T)$.

Step 1. If the tree T is a star or the order of T is no more than 4, then stop algorithm; Otherwise go to Step 2.

Step 2. Find an outside branch vertex v, let $T_v = T - (\Gamma^+(v) \cup \{v\})$. Set $ir(T) = ir(T_v) + |\Gamma^+(v)| - 1$. Replace T by T_v and go to Step 1.

Now, let us analyze the complexity of the above algorithm.

Theorem 2.3 *Let T be a tree with order $n(n \geq 2)$. Then the isolated rupture degree of T can be computed in $O(n^2)$ time.*

Proof. The correctness of this algorithm follows from Lemma 2.1 and Theorem 2.2. It is easy to see that Step 1 can be done in time $O(1)$ in a straightforward manner. In Step 2, we spend $O(n)$ time to search the outside branch vertices of T. Deleting an outside branch vertex v and $\Gamma^+(v)$ will make the order of tree reduce at least 2, so, in Step 2, the above deletion process will carry out at most $\lfloor \frac{n}{2} \rfloor$ time. Hence, Step 2 can be done in time $O(n^2)$. Consequently, the total number of computations required for this algorithm is approximately $O(n^2)$. This completes the proof. □

3. Maximum and minimum isolated rupture degree with given order and maximum degree

In this section, we determine the maximum and minimum isolated rupture degree of trees with given order and maximum degree. In addition, a method for constructing such trees is presented. The following Lemmas are used later.

Lemma 3.1 [15] *For three positive integers m, n $(m \geq n)$ and s, there exists an integer $r(0 \leq r \leq n-1)$ such that $m = sn + r$.*

Lemma 3.2 [9] *Let H be a connected spanning subgraph of a connected graph G. Then $ir(H) \geq ir(G)$.*

Lemma 3.3 [13] *If T is a tree with maximum degree Δ and order n, then T has at most $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ.*

Lemma 3.4 *Let T_1, T_2 be two trees with n vertices. If $\alpha(T_1) \geq \alpha(T_2)$, then $ir(T_1) \geq ir(T_2)$.*

Proof. It is easily seen that a tree T is a bipartite graph. By Lemma 2.3, we know that $ir(T) = 2\alpha(T) - n - 1$. From the fact that $|V(T_1)| = |V(T_2)| = n$, if $\alpha(T_1) \geq \alpha(T_2)$, then we have $ir(T_1) \geq ir(T_2)$. This proof is completed. □

Definition 3.1 [13] A tree is called a *saturated tree* with maximum degree Δ if every vertex has degree equal to either 1 or Δ.

Example 3.1 Consider a tree T with order $n = 14$ and $\Delta = 5$. So we have $\lfloor \frac{n-2}{\Delta-1} \rfloor = \lfloor \frac{14-2}{5-1} \rfloor = 3$, and thus the saturated tree is constructed as follows:

Step 1 Construct a path P_3 of length 2 such that its vertices are labelled as v_1, v_2, v_3 from left to right.

Step 2 Join v_1 to four new vertices v_4, v_5, v_6, v_7, join v_3 to four new vertices v_8, v_9, v_{10}, v_{11}, and then join v_2 to three new vertices v_{12}, v_{13}, v_{14}. The tree is shown in Fig. 2.

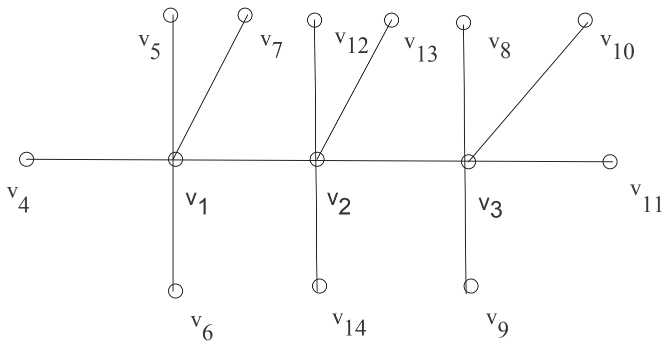

Fig. 2 *A saturated tree with three vertices of degree* $\Delta = 5$.

It is easy to see that the unique tree with order n and maximum degree $\Delta = 2$ is the path P_n, and $ir(P_n)$ was determined by Lemma 2.1. So, in the following, we always assume that $\Delta \geq 3$.

Theorem 3.1 *Let* $T[n, \Delta]$ *be the set of all trees having order* $n (n \geq 4)$ *and maximum degree* $\Delta (\Delta \geq 3)$*. Then*

$$\max_{T \in T[n,\Delta]} ir(T) = \begin{cases} n - 2\left\lfloor \dfrac{n-2}{\Delta-1} \right\rfloor - 1, & \text{if } d\left(\dfrac{n-2}{\Delta-1}\right) < \left\lfloor \dfrac{n-2}{\Delta-1} \right\rfloor \\ n - 2\left\lfloor \dfrac{n-2}{\Delta-1} \right\rfloor - 3, & \text{if } d\left(\dfrac{n-2}{\Delta-1}\right) \geq \left\lfloor \dfrac{n-2}{\Delta-1} \right\rfloor, \end{cases} \tag{3}$$

where $d(\frac{n-2}{\Delta-1})$ *denotes the remainder of* $n - 2$ *divided by* $\Delta - 1$.

Proof. We distinguish two cases:

Case 1. $d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor$.
We construct a tree $T_1 \in T[n, \Delta]$ as follows:
(1) Construct a saturated tree T_0 with $n - d(\frac{n-2}{\Delta-1})$ vertices and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ.

(2) It is easy to see that T_0 has $(\lfloor\frac{n-2}{\Delta-1}\rfloor - 1)$ Δ-edges. Arbitrarily select $d(\frac{n-2}{\Delta-1})$ Δ-edges and subdivide them just once. This gives a tree T_1 with order n that has $\lfloor\frac{n-2}{\Delta-1}\rfloor$ vertices of degree Δ. From the construction we know that $\alpha(T_1) = n - \lfloor\frac{n-2}{\Delta-1}\rfloor$. By Lemma 2.3, we know that

$$ir(T_1) = n - 2\left\lfloor\frac{n-2}{\Delta-1}\right\rfloor - 1.$$

In this case, it is obvious that T_1 is a tree has the maximum independence number in $T[n,\Delta]$. So, by Lemma 3.4, we have $ir(T) \leq ir(T_1)$ for any tree $T \in T[n,\Delta]$, i.e., the tree T_1 constructed above has the maximum isolated rupture degree in $T[n,\Delta]$.

Case 2. $d(\frac{n-2}{\Delta-1}) \geq \lfloor\frac{n-2}{\Delta-1}\rfloor$.
We construct a tree $T_1' \in T[n,\Delta]$ as follows:
(1) Construct a saturated tree T_0' with $n-d(\frac{n-2}{\Delta-1})$ vertices and with $\lfloor\frac{n-2}{\Delta-1}\rfloor$ vertices of degree Δ.
(2) It is easy to see that there exist $(\lfloor\frac{n-2}{\Delta-1}\rfloor - 1)$ Δ-edges in T_0'. First, Subdivide every Δ-edge of T_0' just once to get a new tree T_1' with $n - d(\frac{n-2}{\Delta-1}) + \lfloor\frac{n-2}{\Delta-1}\rfloor - 1$ vertices. By Lemma 3.1, we know that $d(\frac{n-2}{\Delta-1}) \leq \Delta - 2$. So, we have $d(\frac{n-2}{\Delta-1}) - \lfloor\frac{n-2}{\Delta-1}\rfloor + 1 < \Delta - 1 - \lfloor\frac{n-2}{\Delta-1}\rfloor < \Delta - 1$. Then, we randomly select a leaf $v_0 \in V(T_1')$, and, connect it to a set of $d(\frac{n-2}{\Delta-1}) - \lfloor\frac{n-2}{\Delta-1}\rfloor + 1$ independence vertices. Thus, we get a new tree T_1' with order n and with $\lfloor\frac{n-2}{\Delta-1}\rfloor$ vertices of degree Δ. From the construction we know that $\alpha(T_1') = n - \lfloor\frac{n-2}{\Delta-1}\rfloor - 1$. By Lemma 2.4, we know that

$$ir(T_1') = n - 2\left(\left\lfloor\frac{n-2}{\Delta-1}\right\rfloor\right) - 3.$$

In this case, it is obvious that T_1' is a tree has the maximum independence number in $T[n,\Delta]$. So, by Lemma 3.4, we know that for any tree $T \in T[n,\Delta]$, we have $ir(T) \leq ir(T_1')$, i.e., the tree T_1' constructed above has the maximum isolated rupture degree in $T[n,\Delta]$. This completes the proof. □

It is easy to see that Theorem 3.1 gives the methods of constructing trees with maximum isolated rupture degree when their order and maximum degree are given. Moreover, from the proof of Theorem 3.1, we also find that such trees are not unique. Now we give some examples.

Example 3.2 Denote by $T[n,\Delta]$ the set of trees of order n and maximum degree Δ.
(1) Construct a tree T with order 15 and $\Delta = 4$, such that T has the maximum isolated rupture degree in $T[15,4]$. Since $n = 15$, $\Delta = 4$ and $d(\frac{n-2}{\Delta-1}) = d(\frac{13}{3}) = 1 < 4 = \lfloor\frac{13}{3}\rfloor = \lfloor\frac{n-2}{\Delta-1}\rfloor$, such a tree is contained in Fig. 3.
(2) Construct a tree T with order 21 and $\Delta = 6$, such that T has the maximum isolated rupture degree in $T[21,6]$. Since $n = 21$, $\Delta = 6$ and $d(\frac{n-2}{\Delta-1}) = d(\frac{19}{5}) = 4 > 3 = \lfloor\frac{19}{5}\rfloor = \lfloor\frac{n-2}{\Delta-1}\rfloor$, such a tree is contained in Fig. 4.

It is well-known that any connected graph has a spanning tree with the same maximum degree as the graph itself. Thus, we have the following Corollary.

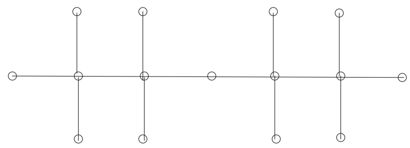

Fig. 3 *A tree with the maximum isolated rupture degree in* $T[15, 4]$.

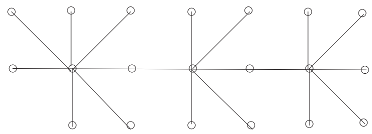

Fig. 4 *A tree with the maximum isolated rupture degree in* $T[21, 6]$.

Corollary 3.1 *If* $G = (V, E)$ *is a connected graph with the order* $|V(G)| = n \geq 4$ *and* $\Delta(G) = \Delta \geq 3$. *Then we have*

$$ir(G) \leq \begin{cases} n - 2\left\lfloor \frac{n-2}{\Delta - 1} \right\rfloor - 1, & \text{if } d\left(\frac{\mathrm{n}-2}{\Delta-1}\right) < \left\lfloor \frac{\mathrm{n}-2}{\Delta-1} \right\rfloor \\ n - 2\left\lfloor \frac{n-2}{\Delta - 1} \right\rfloor - 3, & \text{if } d\left(\frac{\mathrm{n}-2}{\Delta-1}\right) \geq \left\lfloor \frac{\mathrm{n}-2}{\Delta-1} \right\rfloor \end{cases},$$

where $d(\frac{n-2}{\Delta-1})$ *denotes the remainder of* $n - 2$ *divided by* $\Delta - 1$.

Proof. By Lemma 3.2 we know that $ir(G) \leq ir(T)$, where T is a spanning tree of G with $\Delta(G) = \Delta(T)$. It follows from Theorem 3.1 that

$$ir(G) \leq ir(T) \leq \begin{cases} n - 2\left\lfloor \frac{n-2}{\Delta - 1} \right\rfloor - 1, & \text{if } d\left(\frac{\mathrm{n}-2}{\Delta-1}\right) < \left\lfloor \frac{\mathrm{n}-2}{\Delta-1} \right\rfloor \\ n - 2\left\lfloor \frac{n-2}{\Delta - 1} \right\rfloor - 3, & \text{if } d\left(\frac{\mathrm{n}-2}{\Delta-1}\right) \geq \left\lfloor \frac{\mathrm{n}-2}{\Delta-1} \right\rfloor \end{cases},$$

where $d(\frac{n-2}{\Delta-1})$ denotes the remainder of $n-2$ divided by $\Delta-1$. This completes the proof. □

Theorem 3.2 *Let* $T[n, \Delta]$ *be the set of all trees of order* n $(n \geq 4)$ *and maximum degree* Δ $(\Delta \geq 3)$. *Then*

$$\min_{T \in T[n,\Delta]} ir(T) = \begin{cases} \Delta - n - 1, & \text{if } n \leq 2\Delta - 2 \\ 0, & \text{if } n \text{ is odd and } n \geq 2\Delta - 1 \\ -1, & \text{if } n \text{ is even and } n \geq 2\Delta - 1. \end{cases}$$

Proof. We distinguish two cases:

Case 1. $n \leq 2\Delta - 2$.

It is easily seen that $\Delta \geq n-\Delta+2 > n-\Delta-1$. By subdividing $n-\Delta-1$ number of leaf-edges just once in $K_{1,\Delta}$, we get a new tree $T' \in T[n,\Delta]$. Let S be set of added new vertices in subdivision process. Then, $|S| = n-\Delta-1$. For convenience, let $S = \{v_1, v_2, \cdots, v_{n-\Delta-1}\}$. By the definition, we know that all $v_i \in S (i = 1, 2, \cdots, n-\Delta-1)$ are outside branch vertices, and $T' - \bigcup_{i=1}^{n-\Delta-1}(\Gamma^+(v_i) \cup \{v_i\}) = K_{1,2\Delta-n+1}$. By Lemma 2.1 and Theorem 2.1, we have

$$ir(T') = ir(K_{1,2\Delta-n+1}) = 2\Delta - n - 1.$$

In the next, we will prove that for any $T \in T[n,\Delta]$ of the order $n \leq 2\Delta-2$, we have $ir(T) \geq 2\Delta-n-1$. In fact, by Lemma 2.3, we have $ir(T) = 2\alpha(T)-n-1$ for any $T \in T[n,\Delta]$. On the other hand, from the structure of $T \in T[n,\Delta]$, we know that $\alpha(T) \geq \Delta$, so we have $ir(T) \geq 2\Delta - n - 1$.

Case 2. $n \geq 2\Delta - 1$.
By Lemma 2.3, it is easily seen that for any tree T, $ir(T)$ and $|V(T)|$ have the different parity. And by Lemma 2.2, we know that $ir(T) \geq -1$. So, if n is even, then $ir(T) \geq -1$, otherwise $ir(T) \geq 0$. Let T_1 be a tree obtained by subdividing $\Delta - 1$ leaf-edges of the star $K_{1,\Delta-1}$ just once, and then identifying one end vertex of the path $P_{n-2\Delta+2}$ with the vertex $v \in T_1$ whose degree $d_{T_1}(v) = \Delta - 1$. Thus, we get a new tree $T_2 \in T[n,\Delta]$. By Lemma 2.1 and Theorem 2.2, we know that

$$ir(T_2) = ir(P_{n-2\Delta+2}) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ -1, & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. $\square$

4. Isolated rupture degree of gear graphs

Geared systems are used in dynamic modelling. These are graph theoretic models that are obtained by using gear graphs. Similarly the Cartesian product of gear graphs, the complement of a gear graph can be used to design a gear network. We know that the isolated rupture degree is a reasonable parameter to measure the vulnerability of networks. Consequently, these considerations motivated us to investigate the isolated rupture degree of gear graphs. Now we give the following definitions.

Definition 4.1 [2] The *wheel graph* with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle.

Definition 4.2 [3] The *gear graph* G_n is a graph obtained from the wheel graph W_n by subdividing each edge of the outer n-cycle of the W_n just once.

It is easily seen that the gear graph G_n has $2n+1$ vertices and $3n$ edges. In Fig. 5 we display G_6 and we call the vertex u *center vertex* of G_n. Now we give the isolated rupture degree of a gear graph.

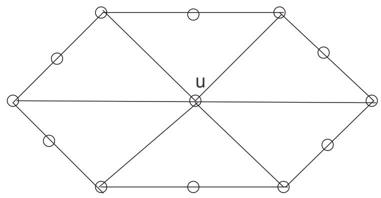

Fig. 5 *Graph G_6.*

Theorem 4.1 *Let G_n be a gear graph. Then $ir(G_n) = 0$.*

Proof. It is easily seen that G_n is a connected bipartite graph. $|V(G_n)| = 2n+1$, and $\alpha(G_n) = n+1$. By Lemma 2.3, we know that

$$ir(G_n) = 2\alpha(G_n) - |V(G_n)| - 1 = 0.$$

The proof is completed. □

In the next, we will study the isolated rupture degree of complement of gear graph G_n. Firstly, we introduce the concept of the *complement* of a graph.

Definition 4.3 [2] The *complement* of a graph G is a graph G^c on the same vertices such that two vertices of G^c are adjacent if and only if they are not adjacent in G.

Theorem 4.2 *Let G_n be a gear graph. Then $ir(G_n^c) = 2 - 2n$.*

Proof. We know that a gear graph G_n can be constructed from a wheel graph W_n by subdividing each edge of the outer n-cycle of the W_n just once. Let S' be a set of vertices of the outer n-cycle in W_n, and let S'' be a set of vertices which are added to the outer n-cycle in G_n. Let u be the center vertex. Since S' and $S'' \cup \{u\}$ are two independent sets in G_n, these vertices form two complete graphs $G_n^c[S']$ and $G_n^c[S'' \cup \{u\}]$ with order n and $n+1$ in G_n^c, respectively. Moreover, each vertex of $G_n^c[S']$ is joined to the vertices of $G_n^c[S'' \cup \{u\}]$ with $n-2$ edges in graph G_n^c. It is obvious that the vertex u in G_n^c is not adjacent to any vertex in $G_n^c[S']$. Let S be a vertex cut set of G_n^c. By the definition of vertex cut set of incomplete graph, we know that $n \le |S| \le 2n-1$. If we remove all the vertices of S from G_n^c, then the number of remaining components is exactly 2. So we have two cases:

Case 1. $n \le |S| \le 2n-2$. There exist two component of $G_n^c - S$, and one of them is of order 1, $m(G_n^c - S) = 2n - |S|$. Thus, we have

$$i(G_n^c - S) - |S| - m(G_n^c - S) = 1 - 2n.$$

Case 2. $|S| = 2n-1$. The two components of $G_n^c - S$ are both of order 1. Then $m(G_n^c - S) = 1$. And so

$$i(G_n^c - S) - |S| - m(G_n^c - S) = 2 - 2n.$$

Thus,

$$ir(G_n^c) = max\{1 - 2n, 2 - 2n\} = 2 - 2n.$$

The proof is completed. □

Now we consider the Cartesian product of two graphs.

Definition 4.4 [2] The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows: $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2 = v_2$.

We note that if G_1 and G_2 are connected, then $G_1 \times G_2$ is connected. And it is well known that, if G is a bipartite graph with bipartition $[A, B]$ and H is bipartite graph with bipartition $[C, D]$, then the Cartesian product of these two bipartite graphs G and H, $G \times H$ is a bipartite graph with bipartition $[(A \times C) \cup (B \times D), (A \times D) \cup (B \times C)]$.

Theorem 4.3 *Let G_n be a gear graph. Then $ir(K_2 \times G_n) = -1$.*

Proof. Since K_2 and G_n are two bipartite graphs, $K_2 \times G_n$ is a connected bipartite graph with $V(K_2 \times G_n) = 4n + 2$, and $\alpha(K_2 \times G_n) = 2n + 1$. By Lemma 2.3, we have

$$ir(K_2 \times G_n) = -1.$$

The proof is completed. □

Theorem 4.4 *Let $m(\geq 3)$ and $n(\geq 3)$ be two positive integers. Then*

$$ir(G_m \times G_n) = 0.$$

Proof. G_m and G_n are two bipartite graphs. Hence, $G_m \times G_n$ is an incomplete connected bipartite graph with $|V(G_m \times G_n)| = (2m+1)(2n+1)$, and $\alpha(G_m \times G_n) = 2mn + m + n + 1$. By Lemma 2.3, we know that

$$ir(G_m \times G_n) = 2\alpha(G_m \times G_n) - |V(G_m \times G_n)| - 1 = 0.$$

The proof is completed. □

5. Some other results on isolated rupture degrees of graphs

In this section, we determine the isolated rupture degree of a the Cartesian product of two special graphs and a special permutation graph. The concept of a permutation graph was introduced by Chartrand and Harary in [4]. It is well known that permutation graphs have high connectivity properties. Since then, many parameters on this kind of graphs have been determined, such as connectivity, chromatic number, crossing number, etc.

Definition 5.1 [4] Let G be a graph whose vertices are labelled $v_1, v_2, \cdots, v_n$ and a permutation $\alpha \in S_n$, where S_n is the symmetric group on $\{1, 2, \cdots, n\}$. Then the *permutation graph* $P_\alpha(G)$ is obtained by taking two copies of G, say G_x with vertex set $\{x_1, x_2, \cdots, x_n\}$ and G_y with vertex set $\{y_1, y_2, \cdots, y_n\}$, along with a set of permutation edges joining x_i of G_x and $y_{\alpha(i)}$ of G_y $(i = 1, 2, \cdots, n)$.

Lemma 5.1 *Let G be a bipartite, k-connected, k-regular graph on n vertices. Then $ir(G) = -1$.*

Proof. It is easy to see that $\alpha(G) = \frac{n}{2}$. Thus, by Lemma 2.3 we have

$$ir(G) = 2\alpha(G) - n - 1 = -1.$$

The proof is completed. □

Theorem 5.1 *Let G_1 be a bipartite, n-regular and n-connected graph with p_1 vertices, and G_2 is a bipartite, m-regular and m-connected graph with p_2 vertices. Then $ir(G_1 \times G_2) = -1$.*

Proof. It is obvious that the graph $G_1 \times G_2$ is an $(m+n)$-regular and $(m+n)$-connected bipartite graph with mn vertices. Then, by Lemma 5.1 we have

$$ir(G_1 \times G_2) = -1.$$

The proof is completed. □

The Cartesian product of n graphs $G_1, G_2, \cdots, G_n$, denoted by $G_1 \times G_2 \times \cdots \times G_n$, is defined inductively as the Cartesian product of $G_1 \times G_2 \times \cdots \times G_{n-1}$ and G_n. In particular, the Cartesian product of k copies of K_2, denoted by Q_k, is called a hypercube of dimensional k.

The following two results can be derived from Theorem 5.1 directly.

Theorem 5.2 *Let m and n be two even positive integers. Then $ir(C_n \times C_m) = -1$ and $ir(C_n \times K_2) = -1$.*

Theorem 5.3 *The isolated rupture degree $ir(Q_k)$ of the hypercube Q_k is -1.*

Theorem 5.4 *Let G be a bipartite, k-regular and k-connected graph with partition $[M, N]$ on n vertices. Then, for a permutation $\alpha \in S_n$ satisfies that*

$$\alpha : \begin{cases} M_x \longrightarrow N_y \\ M_y \longrightarrow N_x, \end{cases}$$

we have $ir(P_\alpha(G)) = -1$, where $[M_x, M_y]$ is the partition of the first copy of G, and $[N_x, N_y]$ is the partition of the second copy of G.

Proof. It is easy to verify that the graph $P_\alpha(G)$ is a $(k+1)$-regular and $(k+1)$-connected bipartite graph with partition $[M_x \cup M_y, N_x \cup N_y]$. By Lemma 5.1, we know that

$$ir(P_\alpha(G)) = -1.$$

The proof is completed. □

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