INTEGRATED INFORMATION ASSESSMENT USING GEOMETRIC ALGEBRA

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Abstract: The goal of the paper is to introduce a universal approach for calculating integrated information assessment (IIA) in complex systems by utilizing the geometric product from geometric algebra (GA). Traditional models of consciousness try to explain how neural networks and cognitive processes give rise to a unified conscious experience. Quantum mechanics (QM) could provide a framework for understanding this integration by suggesting that conscious experience arises from entangled states across different system parts. Thanks to the high redundancy of neural networks, it is possible to realize different variants of cognitive processes in parallel and switch between them as needed. This opens up the possibility of hypothetically creating non-separable (not necessarily non-local) entangled models without requiring a quantum environment. The described IIA algorithm is derived from the assessment of entanglement in QM systems using GA. The results are shown on a set of illustrative examples.

Key words: *geometric algebra, quantum mechanics, quantum entanglement, integrated information theory (IIT), quantum systems, complex systems, non-separability, non-locality, geometric algebra transformer, artificial intelligence, natural language processing (NLP), large language model (LLM)*

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1. Introduction

Geometric algebra (GA) is a mathematical framework [1] that extends traditional calculus by incorporating geometric concepts directly into the algebraic system. It provides a unified language for mathematics and physics, allowing for a more intuitive understanding and manipulation of geometric objects.

Integrated information theory (IIT) is a theoretical framework [2] that seeks to explain the nature of consciousness. According to IIT, a system is conscious if it can integrally and irreducibly integrate information. The key principles of IIT are as follows:

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- *Intrinsic Existence*: Consciousness exists from its own perspective. A system with consciousness has intrinsic existence and cannot be fully explained by its individual parts alone.
- *Composition*: Consciousness comprises different components that integrate to form a unified experience.
- *Information*: Each conscious experience is specific and distinct from other potential experiences.
- *Integration*: Consciousness is not merely a collection of independent components, but a holistic entity in which all parts work together.
- *Definiteness*: It exists at a specific level of integration and does not simultaneously spread across different levels.

Our approach aims not to replace the entire IIR methodology but to demonstrate new possibilities for evaluating the separability of complex systems.

Among other things, quantum mechanics (QM) introduces the concept of entanglement [3], which involves particles being correlated in ways that classical physics cannot explain. The intersection of quantum physics, neuroscience, and cognitive science encourages interdisciplinary collaboration. This collaboration could lead to novel theoretical models and experimental approaches, pushing the boundaries of the understanding of both QM and cognitive science. Some researchers propose that quantum entanglement and superposition could play a role in the integration of information [4], which is a key aspect of IIT.

The paper is structured in such a way that Section 2 summarizes the basic principles of geometric algebra (GA). Section 3 deals with models of quantum informatics, specifically the analysis of the feature known as quantum entanglement. In Section 4, the algorithm for integrated information assessment (IIA) is introduced and Section 5 presents some illustrative examples. Section 6 concludes the paper.

2. The Principles of Geometric Algebra

Geometric algebra (GA) introduces multivectors [1], which extend beyond simple vectors to include scalars (just magnitudes), bivectors (areas), trivectors (volumes), and higher-dimensional elements. A *blade* can be thought of as an oriented segment of a geometric entity, such as a directed line or a plane segment. Rotors are special multivectors that provide rotations in any dimension. They simplify the description of rotations by avoiding the complexity of matrices or quaternions, making the rotation process more intuitive. In GA, it could be combined dimensions in a coherent way. For example, by combining two vectors in three-dimensional space, a plane can be obtained, represented by a bivector.

GA offers a rich and powerful language that unifies various mathematical disciplines, making it easier to describe geometric or physical concepts. It extends beyond traditional algebra by incorporating geometric intuition directly into the algebraic framework, enabling more natural and efficient problem-solving across mathematics and physics.

Consider any two vectors $\vec{a}, \vec{b}$ in $\mathbb{C}^m$ written in orthonormal basis $\{\hat{e}_i\}_{i=1}^m$. The geometric product of two higher dimensional vectors $\vec{a}\vec{b}$ is the fundamental operation in geometric algebra, as it extends traditional algebraic operations:

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}. \tag{1}$$

The dot product $\vec{a} \cdot \vec{b}$ measures how much one vector projects onto another. Additionally, the wedge product $\vec{a} \wedge \vec{b}$, also known as the exterior or outer product, measures the area or higher-dimensional content spanned by two or more vectors. While the dot product focuses on scalar projection and alignment, the wedge product expands the ability to describe and work with areas, volumes, and higher-dimensional geometric constructs. Together, these operations form a robust framework for exploring a wide range of mathematical and physical problems.

Consider two vectors $\vec{a}, \vec{b}$ in coordinates $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$\begin{aligned} \vec{a} &= \hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3, \\ \vec{b} &= 4\hat{e}_1 + 5\hat{e}_2 + 6\hat{e}_3. \end{aligned} \tag{2}$$

Their wedge product is (using the anti-commutativity $\hat{e}_i \wedge \hat{e}_j = -\hat{e}_j \wedge \hat{e}_i$ and the fact that $\hat{e}_i \wedge \hat{e}_i = 0$):

$$\begin{aligned} \vec{a} \wedge \vec{b} &= (\hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3) \wedge (4\hat{e}_1 + 5\hat{e}_2 + 6\hat{e}_3) = \\ &= \hat{e}_1 \wedge 4\hat{e}_1 + \hat{e}_1 \wedge 5\hat{e}_2 + \hat{e}_1 \wedge 6\hat{e}_3 + 2\hat{e}_2 \wedge 4\hat{e}_1 + 2\hat{e}_2 \wedge 5\hat{e}_2 + 2\hat{e}_2 \wedge 6\hat{e}_3 + \\ &+ 3\hat{e}_3 \wedge 4\hat{e}_1 + 3\hat{e}_3 \wedge 5\hat{e}_2 + 3\hat{e}_3 \wedge 6\hat{e}_3 = -3\hat{e}_1 \wedge \hat{e}_2 - 6\hat{e}_1 \wedge \hat{e}_3 - 3\hat{e}_2 \wedge \hat{e}_3. \end{aligned} \tag{3}$$

This non-zero bivector indicates that the vectors $\vec{a}, \vec{b}$ span a non-zero area.

3. Quantum Informatics

3.1 Quantum Models

In quantum informatics [3] two qubits can be defined:

$$|\psi_1\rangle = \alpha_1 \cdot |0\rangle + \beta_1 \cdot |1\rangle, \quad |\psi_2\rangle = \alpha_2 \cdot |0\rangle + \beta_2 \cdot |1\rangle. \tag{4}$$

Bracket notation $|\cdot\rangle$ is a powerful way to represent and manipulate quantum states.

Their superposition can be expressed as:

$$\begin{aligned} |\psi\rangle &= (\alpha_1 \cdot |0\rangle + \beta_1 \cdot |1\rangle) \otimes (\alpha_2 \cdot |0\rangle + \beta_2 \cdot |1\rangle) = \\ &= \alpha_1 \cdot \alpha_2 \cdot |00\rangle + \alpha_1 \cdot \beta_2 \cdot |01\rangle + \beta_1 \cdot \alpha_2 \cdot |10\rangle + \beta_1 \cdot \beta_2 \cdot |11\rangle. \end{aligned} \tag{5}$$

Coefficients α_i, β_j are complex parameters for all i, j. Symbol $\otimes$ represents the tensor product and the probabilistic normalization condition can be given as:

$$|\alpha_1 \cdot \alpha_2|^2 + |\alpha_1 \cdot \beta_2|^2 + |\beta_1 \cdot \alpha_2|^2 + |\beta_1 \cdot \beta_2|^2 = 1. \tag{6}$$

We can assume an equal probability of zeros and ones for all combinations. This refers to probabilistic classical bits, where superposition allows to work simultaneously with all possible combinations. For example, with three probability bits, we have a superposition of eight combinations that each have an equal probability:

$$
\begin{aligned}
|\psi\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) = \\
= \frac{1}{2^{3/2}} \cdot (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) . \qquad (7)
\end{aligned}
$$

This representation of quantum binary systems can generally be extended to more complex n-dimensional space.

3.2 Quantum Entanglement

Quantum entanglement is a fundamental phenomenon in which particles become interconnected in a way that the state of one particle instantly affects the state of another, regardless of the space-time distance between them. Entanglement enables interactions across different dimensions and scales, providing a deeper comprehension of how different parts of a system impact each other. This is particularly important for modeling and analyzing complex systems in various fields such as biology, chemistry, and physics.

Entanglement swapping [5] is a phenomenon employed in quantum networks to expand the reach of entanglement. By entangling pairs of particles at intermediate nodes, it becomes possible to create long-distance entanglement, which is crucial in constructing scalable quantum complex systems. If we have four events, with the first and second events being entangled, and the third and fourth events also being entangled, then as soon as there is entanglement between the first and third events, the second and fourth events will also become entangled without any exchange of information between them. It is important to note that these phenomena can occur even when there is a significant amount of distance in both space and time between them.

Entanglement entropy [6] is used to quantify the amount of entanglement in a quantum system and measures the information shared between its different parts. When the subsystems are maximally entangled, the entanglement entropy reaches its maximum value. Consider two entangled qubits. When observing a single qubit in isolation, its state may seem random due to its entanglement with the other qubit. The entanglement entropy quantifies this randomness by reflecting the amount of information connecting the state of one qubit to the state of the other.

Bell states [3] are maximally entangled, meaning the highest degree of entanglement between two qubits:

$$
\begin{aligned}
\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) , \quad \left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) , \\
\left|\Psi^+\right\rangle &= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) , \quad \left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) . \qquad (8)
\end{aligned}
$$

If there is no entanglement, the entanglement entropy is zero. By measuring the information shared between subsystems, entanglement entropy enables a deeper understanding of the complexity and coherence of entangled subsystems.

4. Integrated Information Assessment

In quantum mechanics (QM), observables correspond to measurements that can be performed on the system. In geometric algebra (GA), these measurements are represented by specific multivectors or combinations of multivectors. The outcome of a measurement can be interpreted as the projection of the state multivector onto the observable multivector, thus providing a geometric understanding of quantum measurement. While quantum entanglement and the wedge product originate from different areas of science, they share a fundamental focus on integration, multidimensional interactions, and holistic descriptions.

4.1 Decomposition of Quantum Binary Systems

For simplicity, we will analyze an n-dimensional quantum system superposed only from qubits that have a value of either zero or one. This simplification is a logical first approximation for modeling neural networks, where an active neuron is represented by one (indicating it fires) and a passive neuron by zero (indicating it doesn't fire). The extension to three dimensions (qutrit) and higher dimensions (qudit, etc.) are discussed in [3, 7].

The studied quantum system encompasses all possible combinations generally captured by the complex parameters $\alpha_{k_1,k_2,\ldots,k_n}$:

$$|\Psi\rangle = \sum_{k_1,k_2,\ldots,k_n=0}^{1} \alpha_{k_1,k_2,\ldots,k_n} |k_1, k_2, \ldots, k_n\rangle \,, \tag{9}$$

where $k_1, k_2, \ldots, k_n \in \{0, 1\}$.

The quantum system Eq. (9) can be divided into two parts. The first part $|k_1, k_2, \ldots, k_m\rangle$ represents the input information (causal trigger, cause-effect), while the second part is the result of the operation (post-measurement, consequence of the input information):

$$\begin{aligned} |\Psi\rangle &= \sum_{k_1,k_2,\ldots,k_n=0}^{1} \alpha_{k_1,k_2,\ldots,k_n} |k_1, k_2, \ldots, k_n\rangle = \\ &= \sum_{k_1,k_2,\ldots,k_m=0}^{1} |k_1, k_2, \ldots, k_m\rangle \\ &\quad \left(\sum_{k_{m+1},k_{m+2},\ldots,k_n=0}^{1} \alpha_{k_1,k_2,\ldots,k_n} |k_{m+1}, k_{m+2}, \ldots, k_n\rangle \right) . \end{aligned} \tag{10}$$

We can mark the post-measurement $|p_{k_1,k_2,\ldots,k_m}\rangle$ state as:

$$|p_{k_1,k_2,\ldots,k_m}\rangle = \sum_{k_{m+1},k_{m+2},\ldots,k_n=0}^{1} \alpha_{k_1,k_2,\ldots,k_n} |k_{m+1},k_{m+2},\ldots,k_n\rangle . \tag{11}$$

This model can effectively describe a variety of classic and quantum functions between input and output vectors including the superposition of all variants, even in the presence of forward and backward loops.

It should be noted that the model Eq. (10) and (11) can also be used in reverse order. In this mode, the input information (causal trigger, cause-effect) consists of the outputs of the analyzed system. The second part (post-measurement, consequence of the input information) will then include all possible system inputs corresponding to the specific system outputs. This approach is often used in IIR theory [2] where intrinsic information must be evaluated by perturbing a set of system states in all possible way and not just observing them like in case of Shannon information.

4.2 Entanglement Quantification by Wedge Product

Let us define the multidimensional wedge products:

$$\bigwedge_{i=1}^{n} |v_i\rangle = |v_1\rangle \wedge |v_2\rangle \wedge \cdots \wedge |v_n\rangle \tag{12}$$

The basic idea of integrated information assessment (IIA) is to analyze the entanglement between all possible combinations (pairs, triples and tuples) of the output state $|p_{k_1,k_2,\ldots,k_m}\rangle$. Each set of combinations contributes to the magnitude of entanglement quantification [8] as follows:

$$E_m = f_m \cdot \left\| \bigwedge_{k_1,k_2,\ldots,k_m=0}^{1} |p_{k_1,k_2,\ldots,k_m}\rangle \right\| , \tag{13}$$

$$E_{m-1} = f_{m-1} \cdot \left(\sum_{l_1,l_2,\ldots,l_m=0}^{1} \left\| \bigwedge_{\substack{k_1,k_2,\ldots,k_m=0 \\ k_1\neq l_1,\ldots,k_m\neq l_m}}^{1} |p_{k_1,k_2,\ldots,k_m}\rangle \right\| \right) , \tag{14}$$

$$\cdots$$

$$E_2 = f_2 \cdot \left(\sum_{l_1,l_2,\ldots,l_m=0}^{1} \sum_{k_1,k_2,\ldots,k_m=0}^{1} \| |p_{k_1,k_2,\ldots,k_m}\rangle \wedge |p_{l_1,l_2,\ldots,l_m}\rangle \| \right) , \tag{15}$$

$$E = E_m + E_{m-1} + \cdots + E_2. \tag{16}$$

The weighting constants $f_2, f_3, \ldots, f_m$ can be determined in such a way that the overall value of entanglement quantification is equal to one for the maximal possible entanglement and zero for no entanglement. Parameters $E_2, E_3, \ldots, E_m$ express the evaluation of entanglement for all possible pairs, triples, and m-tices. Parameter E represents the sum of all internal entanglement within the complex system.

4.3 Separability Assessment

Separability in quantum physics refers to the ability to express a quantum system as a tensor product of its individual components. Separability concept ensures that information about the overall system is not lost when breaking it down into its parts. A complex system is thus separable across all bipartitions only if every single bipartition is separable. Therefore, a necessary and sufficient criterion for system separability is $E = 0$.

On the other hand, the concept of non-separability means that a complex system cannot be decomposed into sub-parts without loss of information. In other words, all components of the system are entangled. Therefore, the entanglement evaluated by the wedge product can be regarded as a measure of integrated information in a complex system.

It's important to add that quantum properties like entanglement are not limited to the microworld, where they exhibit non-separability and non-locality, but can also be applied to redundant systems like our brain [9] that possesses high non-separability features.

When the weighting parameters in a quantum system are not complex, we can model classical probabilistic input-output relationships. In a fully deterministic system, the parameters can only take on values of zero or one. A value of one signifies the resulting state determined by inherent logic, while a value of zero indicates the other impossible states. The magnitude of entanglement can be measured using the wedge product algorithm that considers all possible combinations and subsets of different variants. Using the entanglement quantification, we can provide the assessment of integrated information also in classical systems.

5. Examples of Integrated Information Assessment

In the following illustrative examples, we will present the IIA algorithm for both quantum and classical systems.

5.1 Two-dimensional Binary System

Let us define the input quantum qubit:

$$|\phi\rangle_{\mathrm{IN}} = \alpha_0 |0\rangle_{\mathrm{IN}} + \alpha_1 |1\rangle_{\mathrm{IN}} . \tag{17}$$

If the input state is $|0\rangle_{\mathrm{IN}}$, the system output will assign qubit $|\phi_1\rangle_{\mathrm{OUT}}$. However, if the input state is $|1\rangle_{\mathrm{IN}}$, the output assigns $|\phi_2\rangle_{\mathrm{OUT}}$:

$$|\phi_1\rangle_{\mathrm{OUT}} = \beta_0 |0\rangle_{\mathrm{OUT}} + \beta_1 |1\rangle_{\mathrm{OUT}} , \tag{18}$$

$$|\phi_2\rangle_{\mathrm{OUT}} = \gamma_0 |0\rangle_{\mathrm{OUT}} + \gamma_1 |1\rangle_{\mathrm{OUT}} . \tag{19}$$

We can establish the resulting quantum model:

$$\begin{aligned} |\psi\rangle &= |0\rangle_{\mathrm{IN}} \alpha_0 \left(\beta_0 |0\rangle_{\mathrm{OUT}} + \beta_1 |1\rangle_{\mathrm{OUT}}\right) + |1\rangle_{\mathrm{IN}} \alpha_1 \left(\gamma_0 |0\rangle_{\mathrm{OUT}} + \gamma_1 |1\rangle_{\mathrm{OUT}}\right) = \\ &= \alpha_0\beta_0 |00\rangle + \alpha_0\beta_1 |01\rangle + \alpha_1\gamma_0 |10\rangle + \alpha_1\gamma_1 |11\rangle . \end{aligned} \tag{20}$$

Based on Eq. (15) the entanglement quantification can be determined:

$$E_2 \propto |\alpha_0\beta_0\alpha_1\gamma_1 - \alpha_0\beta_1\alpha_1\gamma_0|^2 = |\alpha_0\alpha_1(\beta_0\gamma_1 - \beta_1\gamma_0)|^2 . \tag{21}$$

We can observe that the maximum values can be achieved when the following set of equations are fulfilled:

$$\beta_0\gamma_1 = \pm 1, \beta_1\gamma_0 = 0, \tag{22}$$

or alternatively:

$$\beta_0\gamma_1 = 0, \beta_1\gamma_0 = \pm 1. \tag{23}$$

Both variants Eqs. (22) and (23) correspond to the well-known Bell states Eq. (8), which represent maximal non-separability. The zero value, indicating no entanglement, is reached in the situation that corresponds to the linear dependence:

$$\beta_0\gamma_1 = \beta_1\gamma_0. \tag{24}$$

This means that the system is separable.

5.2 Three-dimensional Binary System

Let us define the 3-dimensional quantum binary system:

$$\begin{aligned}|\psi\rangle = {} & \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle + \\ & + \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle .\end{aligned} \tag{25}$$

First two digits are supposed to be the input vector, third one represents the quantum post-measurement. We can rewrite the quantum model as follows:

$$\begin{aligned}|\psi\rangle &= |00\rangle(\alpha_{000}|0\rangle + \alpha_{001}|1\rangle) + |01\rangle(\alpha_{010}|0\rangle + \alpha_{011}|1\rangle) + \\ &\quad + |10\rangle(\alpha_{100}|0\rangle + \alpha_{101}|1\rangle) + |11\rangle(\alpha_{110}|0\rangle + \alpha_{111}|1\rangle) = \\ &= |00\rangle|p_{00}\rangle + |01\rangle|p_{01}\rangle + |10\rangle|p_{10}\rangle + |11\rangle|p_{11}\rangle ,\end{aligned} \tag{26}$$

$$\begin{aligned}p_{00} &= \alpha_{000}|0\rangle + \alpha_{001}|1\rangle , \\ p_{01} &= \alpha_{010}|0\rangle + \alpha_{011}|1\rangle , \\ p_{10} &= \alpha_{100}|0\rangle + \alpha_{101}|1\rangle , \\ p_{11} &= \alpha_{110}|0\rangle + \alpha_{111}|1\rangle .\end{aligned} \tag{27}$$

Entanglement quantification Eq. (15) can be computed as:

$$\begin{aligned}& E_2 \propto \|p_{00} \wedge p_{01}\| + \|p_{00} \wedge p_{10}\| + \|p_{00} \wedge p_{11}\| + \\ & + \|p_{01} \wedge p_{10}\| + \|p_{01} \wedge p_{11}\| + \|p_{10} \wedge p_{11}\| = \\ = & \left(|\alpha_{000}\alpha_{011} - \alpha_{001}\alpha_{010}|^2 + |\alpha_{000}\alpha_{101} - \alpha_{001}\alpha_{100}|^2 + |\alpha_{000}\alpha_{111} - \alpha_{001}\alpha_{110}|^2\right) + \\ & + \left(|\alpha_{010}\alpha_{101} - \alpha_{011}\alpha_{100}|^2 + |\alpha_{010}\alpha_{111} - \alpha_{011}\alpha_{110}|^2 + |\alpha_{100}\alpha_{111} - \alpha_{101}\alpha_{110}|^2\right) .\end{aligned} \tag{28}$$

As an illustrative practical example, we can analyze separability of classical and deterministic XOR-gate with two inputs and one output. The bracket expression is given:

$$\begin{aligned} & |\psi\rangle_{\mathrm{XOR}} \propto |001\rangle + |010\rangle + |100\rangle + |111\rangle = \\ = \; & |00\rangle\,(0\cdot|0\rangle + 1\cdot|1\rangle) + |01\rangle\,(1\cdot|0\rangle + 0\cdot|1\rangle) + |10\rangle\,(1\cdot|0\rangle + 0\cdot|1\rangle) \\ & + |11\rangle\,(0\cdot|0\rangle + 1\cdot|1\rangle)\,. \end{aligned} \tag{29}$$

The symbol $\propto$ means equality up to a constant. Post-measurement vectors could be defined:

$$p_{00} = 0\cdot|0\rangle + 1\cdot|1\rangle\,, p_{01} = 1\cdot|0\rangle + 0\cdot|1\rangle\,, p_{10} = 1\cdot|0\rangle + 0\cdot|1\rangle\,, p_{11} = 0\cdot|0\rangle + 1\cdot|1\rangle\,. \tag{30}$$

The entanglement quantification (15) is computed as:

$$\begin{aligned} & E_{2,\mathrm{XOR}} \propto \|p_{00}\wedge p_{01}\| + \|p_{00}\wedge p_{10}\| + \|p_{00}\wedge p_{11}\| + \|p_{01}\wedge p_{10}\| + \\ & + \|p_{01}\wedge p_{11}\| + \|p_{10}\wedge p_{11}\| = \\ = \; & 1+1+0+0+1+1 = 4. \end{aligned} \tag{31}$$

Let us compare the separability assessment between XOR-gate and OR-gate. The OR-gate model is given as:

$$\begin{aligned} & |\psi\rangle_{OR} \propto |000\rangle + |011\rangle + |101\rangle + |111\rangle = \\ = \; & |00\rangle\,(1\cdot|0\rangle + 0\cdot|1\rangle) + |01\rangle\,(0\cdot|0\rangle + 1\cdot|1\rangle) + |10\rangle\,(0\cdot|0\rangle + 1\cdot|1\rangle) + \\ & + |11\rangle\,(0\cdot|0\rangle + 1\cdot|1\rangle)\,. \end{aligned} \tag{32}$$

The OR-gate entanglement quantification is expressed as:

$$\begin{aligned} & E_{2,\mathrm{OR}} \propto \|p_{00}\wedge p_{01}\| + \|p_{00}\wedge p_{10}\| + \|p_{00}\wedge p_{11}\| + \|p_{01}\wedge p_{10}\| + \\ & + \|p_{01}\wedge p_{11}\| + \|p_{10}\wedge p_{11}\| = \\ = \; & 1+1+1+0+0+0 = 3. \end{aligned} \tag{33}$$

These results imply that both XOR- and OR-gate are non-separable, meaning they cannot be decomposed into sub-functions. This confirms that the information between two inputs and one output are integrated. The numerical calculation indicates that the XOR-gate has a higher degree of integration compared to the OR-gate.

5.3 Four-dimensional Binary System

This example illustrates the extension of a binary system to a three-state system, where states $|0\rangle, |1\rangle, |2\rangle$ are represented as $|00\rangle, |01\rangle, |10\rangle$. Let us suppose that first two digits $|00\rangle, |01\rangle, |10\rangle$ represent the input vector, and third and fourth ones $|00\rangle, |01\rangle, |10\rangle$ are the quantum post-measurements:

$$\begin{aligned} |\psi\rangle \; = \; & |00\rangle\,(\alpha_{0000}\,|00\rangle + \alpha_{0001}\,|01\rangle + \alpha_{0010}\,|10\rangle) + |01\rangle \\ & (\alpha_{0100}\,|00\rangle + \alpha_{0101}\,|01\rangle + \alpha_{0110}\,|10\rangle) + \\ & + |10\rangle\,(\alpha_{1000}\,|00\rangle + \alpha_{1001}\,|01\rangle + \alpha_{1010}\,|10\rangle) = \\ = \; & |00\rangle\,|p_{00}\rangle + |01\rangle\,|p_{01}\rangle + |10\rangle\,|p_{10}\rangle\,, \end{aligned} \tag{34}$$

$$\begin{aligned}
p_{00} &= \alpha_{0000}\,|00\rangle + \alpha_{0001}\,|01\rangle + \alpha_{0010}\,|10\rangle\,, \\
p_{01} &= \alpha_{0100}\,|00\rangle + \alpha_{0101}\,|01\rangle + \alpha_{0110}\,|10\rangle\,, \\
p_{10} &= \alpha_{1000}\,|00\rangle + \alpha_{1001}\,|01\rangle + \alpha_{1010}\,|10\rangle\,.
\end{aligned} \tag{35}$$

Entanglement quantification can be computed for this reduced four-dimensional model as:

$$\begin{aligned}
& E_3 \propto \|p_{00} \wedge p_{01} \wedge p_{10}\| = \det \begin{vmatrix} \alpha_{0000} & \alpha_{0001} & \alpha_{0010} \\ \alpha_{0100} & \alpha_{0101} & \alpha_{0110} \\ \alpha_{1000} & \alpha_{1001} & \alpha_{1010} \end{vmatrix}^2 = \\
= \; & |\alpha_{0000}\left(\alpha_{0101}\alpha_{1010} - \alpha_{0110}\alpha_{1001}\right) - \alpha_{0001}\left(\alpha_{0100}\alpha_{1010} - \alpha_{0110}\alpha_{1000}\right) + \\
& + \alpha_{0010}\left(\alpha_{0100}\alpha_{1001} - \alpha_{0101}\alpha_{1000}\right)|^2\,.
\end{aligned} \tag{36}$$

$$\begin{aligned}
& E_2 \propto \|p_{00} \wedge p_{01}\| + \|p_{00} \wedge p_{10}\| + \|p_{01} \wedge p_{10}\| = \\
= \; & \Big(|\alpha_{0001}\alpha_{0110} - \alpha_{0010}\alpha_{0101}|^2 + |\alpha_{0010}\alpha_{0100} - \alpha_{0000}\alpha_{0110}|^2 + \\
& + |\alpha_{0000}\alpha_{0101} - \alpha_{0001}\alpha_{0100}|^2\Big) + \\
+ \; & \Big(|\alpha_{0001}\alpha_{0101} - \alpha_{0010}\alpha_{1001}|^2 + |\alpha_{0010}\alpha_{1000} - \alpha_{0000}\alpha_{1010}|^2 + \\
& + |\alpha_{0000}\alpha_{1001} - \alpha_{0001}\alpha_{1000}|^2\Big) + \\
+ \; & \Big(|\alpha_{0101}\alpha_{1010} - \alpha_{0110}\alpha_{1001}|^2 + |\alpha_{1000}\alpha_{0110} - \alpha_{0100}\alpha_{1010}|^2 + \\
& + |\alpha_{0100}\alpha_{1001} - \alpha_{0101}\alpha_{1000}|^2\Big)
\end{aligned} \tag{37}$$

The given examples can be generalized to include much more complex structure created by combining individual components [10] that can be ordered either sequentially or in parallel, and also in backward and forward loops. The superposition of all possible combinations may encompass not only past and present states but also future states (causes and effects partition) that are linked to studied subsystem, as noted in IIT [2].

When it comes to more and more qubits, we can use the wedge product to assess these vectors and investigate their entanglement properties [17]. In order for the qubits to be entangled, it is necessary that their combined state cannot be factored into the product of two separate states. In terms of geometric algebra, this implies that the wedge product should result in a non-trivial multivector (such as a bivector or higher).

6. Conclusion

The paper presents a GA-based algorithm for measuring integrated information in complex systems. The IIA algorithm relies on the assumption that QM is capable to describe a classical complex system, where the weighting coefficients could be real rather than complex. Additionally, the bracket notation can represent all the potential combinations and relations between past, current, future inputs and outputs.

In the paper, we also explored the geometric aspect of enhancing the traditional QM approach by incorporating visually distinct forms of quantum entanglement. This approach allows for a clearer understanding of how quantum entanglement affects the behavior of a complex system including higher dimensional components with their multidimensional entanglement.

Entangled states can be alternatively expressed by multivectors in GA, which can effectively capture the complex links between sets of two, three, up to n-quantum components. The wedge product in GA integrates the information of individual vectors into a combined geometric entity, such as a plane or volume. By using this approach, it becomes possible to better quantify integrated information of classical or quantum complex systems on different resolution levels.

The resulting methodology can also provide better explanation of emergent phenomena. In a GA, the individual multi-dimensional object has either a positive or negative orientation. This characteristic is necessary for modeling high-dimensional resonances that emerges at various resolution levels between different subsystems. When it comes to positive or negative multi-dimensional object orientations in GA, we can deduce that as we add more and more system components, the complex system gradually gains a better model of its behavior trajectory.

This phenomenon can be likened to QM concept known as Feynman's integral through trajectories [14] revealing that all possible QM paths ultimately align with the established laws of classical physics. In a similar way, it is possible to approach to the information systems and use the wave probabilistic functions [15] to capture emergent phenomena [16], especially in soft systems.

The artificial intelligence (AI) with natural language processing (NLP) uses the knowledge representation in a multidimensional vector space. Large language models (LLMs) are highly advanced NLP models that have been trained on extensive quantities of text data in order to comprehend, generate, and manipulate natural language. These models utilize deep learning techniques, specifically transformer architectures [12], to carry out a diverse array of language-related tasks.

Transformers estimate the context of a given text using various sets of vectors that abstract the knowledge. For example, if one vector represents the city Berlin and another city Tokyo, it is reasonable to assume that similar spatial attributes would be associated with the representation of national dishes like bratwurst or sushi, even though they exist in different vector subspaces. Currently used Transformers work with a vector dot product to determine the context attention matrix. These transformers could be replaced with a more general geometric product to introduce geometric algebra transformers (GATr) [13].

The use of GA can also be observed in how it incorporates the orientation of individual geometric objects (plus/minus signs as for complex numbers in QM [11]). This orientation can be modified through different mathematical functions, either by summing up or subtracting it. Furthermore, when working with multidimensional geometric objects, the operations performed can impact the shape of objects with lower dimensions. This enables the modeling of emergence properties (information resonances) in complex systems.

In the context of cognitive processes, the geometric algebra can serve as a tool for synthesizing information from different sources. If we have two different sensations (e.g., visual and auditory stimuli) represented by vectors their wedge product could represent an integrated sensation (a new quality) that contains both visual and auditory components.

So far, we have only discussed in detail the geometric product. However, GA includes several other operations. The *join* operation, denoted by the symbol $\vee$, combines two geometric objects to form a new one that includes all the points from both of the original objects. This operation enables the creation of the smallest geometric object that encompasses both of the input objects. The *meet* operation is a powerful tool for finding intersections of subspaces. It relies on the geometric product and the concept of duality to compute these intersections. The use of GA tools, allows for the manipulation and composition of individual geometric objects (using positive and negative signs) and enables the expansion to information resonance explained in relation to human analytical and synthetic thinking [16] into higher dimensions.

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