ASYMMETRIC ORIENTEERING PROBLEM WITH PROFITABLE PENALTY

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Abstract: This paper solves a modified version of the asymmetric traveling salesman problem with the possibility of omitting certain nodes and with a defined time limit for the total travel time, also referred to as the asymmetric orienteering problem (AOP). This problem belongs to the class of NP-hard problems. A proposed mathematical model maximizes the total score gained from visiting nodes within a predefined time limit. The possibility of exceeding the time limit, which results in a penalty to the total score, is also considered. The profitable penalty is examined, i.e., whether accepting the penalty can be advantageous for increasing the total score. The problem is demonstrated in a case study from the ski adventure race, organized in the Jizera Mountains in the Czech Republic.

Key words: *AOP with penalty, profitable penalty, ski adventure race*

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1. Introduction

Distribution problems represent a broad category of optimization challenges with wide-ranging practical applications. One significant subset of distribution problems is routing problem, where the goal is to determine the optimal routes from one or more depots (or origins) to a set of nodes (cities, customers, or locations). At the core of routing problems lies the traveling salesman problem (TSP). A more recent variant of the TSP is the TSP with profits, where the traveler must complete their journey within specified constraints such as time, cost, or distance, while optimizing a given objective. Unlike the traditional TSP, this variant does not require visiting all places. The most widely studied example of TSP with profits is the orienteering problem (OP), which originated from an outdoor sport where participants navigate between control points in a set time limit. In the OP, the traveler earns a profit or reward from each visited node, and the goal is to maximize the total profit within

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the available time. This variant is also known as the selective traveling salesman problem. The asymmetric orienteering problem (AOP) is a newer variant of the classical TSP that introduces additional constraints and complexities. The key characteristic of the problem is the asymmetric nature of the matrix of unproductive crossings between nodes, i.e., the distances or costs per trip between two nodes may differ based on the path's direction. Additionally, there is a possibility of omitting some nodes for various reasons, e.g., due to the existence of a time limit allocated for node visits, low efficiency of serving some nodes, or other logistical constraints. The time limit adds a dynamic element to the task making it more realistic in practical applications where time management is critical.

The optimization objectives can vary based on the specific practical application. These objectives might include maximizing total profit within a predetermined time, minimizing total distance, minimizing travel time, or striking a balance between distance and time while considering the omission of certain nodes.

This paper focuses on the asymmetric orienteering problem (AOP) as a specialized extension of the orienteering problem (OP). Assuming that each visited node is assigned a score and our goal is to maximize the total score while accounting for the asymmetry of the problem, the time limit, and the necessity of selecting only certain nodes, an additional dynamic element has been introduced in the form of a penalty for exceeding the time limit. However, we may still achieve a higher overall score by exceeding the predefined time limit, even when factoring in the penalty (i.e., the profitable penalty). This is the aim of this paper and a case study from the sport discipline of Ski Adventure Race in the Czech Republic serves as an example to demonstrate the model and optimization results.

2. Literature Review

The orienteering problem is also known by other names, such as the selective traveling salesperson problem [7, 16], the maximum collection problem [3, 15] and the bank robber problem [1]. It belongs to the routing problems and is close to the traveling salesman problem (TSP) and the vehicle routing problem (VRP). The OP is a well-known routing problem where the objective is to maximize the total score collected by visiting a subset of locations within a given time or distance constraint. On the other hand, VRP typically involves optimizing routes for a fleet of vehicles to service all customers, minimizing the total distance or cost. Unlike OP, VRP often requires that all nodes (customers) are visited, and the focus is on optimizing routes for efficiency in terms of distance, time, or cost. The OP maximizes the total score by visiting a subset of locations within a time or distance limit, while the TSP aims to find the shortest route that visits all locations and returns to the starting point. In OP, not all nodes need to be visited, whereas in TSP, every node must be visited. The key difference is that OP optimizes the profit under constraints, while TSP optimizes the minimum distance without constraints. Surveys on the TSP with profits [5] and on Hamiltonian and non-Hamiltonian problems [18] position the OP among other routing problems, both with and without profits, and highlight the distinctions. Both papers [5, 18] briefly address the OP, discussing some solution strategies and a few of its extensions and variants.

The book chapter titled "The Generalized Traveling Salesman and Orienteering Problems" [6] explores advanced variations of the classic TSP, explicitly focusing on the generalized traveling salesman problem (GTSP) and the OP. In this chapter, the authors thoroughly examine these complex combinatorial optimization problems. The GTSP extends the traditional TSP by requiring the selection of one node from each of several predefined groups or clusters while minimizing the total travel distance. The OP, on the other hand, involves maximizing the total score collected from visiting a subset of locations within a given time or distance limit. The chapter discusses the practical applications and challenges associated with GTSP and OP, offering insights into how these variations can be effectively addressed using optimization methods.

Detailed description and mathematical models of all these models can be found in the book "Orienteering problems: Models and algorithms for vehicle routing problems with profits" [21].

Similar to OP and TSP, an asymmetric alternative to these problems exists. the asymmetric orienteering problem (AOP) focuses on maximizing the total score by visiting a subset of nodes within a given time or distance limit, where travel costs between nodes can vary based on direction. In contrast, the asymmetric traveling salesman problem (ATSP) seeks to minimize the total travel distance or cost while requiring that all nodes be visited, also with asymmetric travel costs [19].

The OP has gained significant attention over the past few decades. Several well-known variants have been extensively studied, including the team OP [4], the capacitated team OP [11], the (team) OP with time windows, and the time-dependent OP [21]. More recently, a range of new OP variants has emerged, such as the stochastic OP, the generalized OP, the arc OP, the multi-agent OP [22], and the clustered OP [12], among others.

A comprehensive overview of the OP covering the history, various formulations, and the different variants of the OP can be found in the paper titled "The Orienteering Problem: A Survey" [22] and in the book "Orienteering problems: Models and algorithms for vehicle routing problems with profits" [21]. The authors highlight the practical applications of the OP in fields such as tourism, logistics, and other areas where optimal route planning is critical.

Another paper titled "Orienteering Problem: A Survey of Recent Variants, Solution Approaches, and Applications" [9] also provides a comprehensive survey of the OP and discusses the latest applications, including the tourist trip design problem and the mobile crowdsourcing problem. The paper categorizes the variants based on different constraints and objectives, such as multi-period OP, team OP and time-dependent OP. For each variant, the authors discuss its specific challenges and the corresponding solution approaches, including exact algorithms, heuristics, and metaheuristics.

The basic team OP, a formulation often used in logistics and operations research, involves a fleet of vehicles tasked with completing operations such as deliveries or collections while maximizing overall profit [4, 21]. The problem requires coordinating the vehicles so they work together efficiently, adhering to constraints such as time limits for each vehicle. The goal is to divide tasks and optimize routes while balancing profitability and operational feasibility.

The practical use of OP and AOP is extensive based on the literature [6, 8–10, 21]. Companies such as FedEx, UPS, and Amazon use route optimization to minimize time and distance when delivering packages. The ability to skip less important stops due to time constraints or dynamic changes in delivery priorities can increase efficiency. Infrastructure maintenance and repair companies can optimize the routes of their technicians, and in cases of time pressure, it may be advisable to skip non-essential maintenance. Home health care providers can optimize the routes of nurses or caregivers visiting patients, considering patient priority and travel time. Vehicles in integrated rescue systems (ambulances, police, and fire trucks) must reach critical locations as quickly as possible. In disaster scenarios, some locations may need to be skipped to prioritize high-urgency calls within a limited amount of time. Optimizing police patrol routes is also important to cover high-crime areas more frequently while skipping lower-risk areas if time does not allow full coverage. Companies optimizing itineraries for visits by multiple suppliers or partners can use AOP to minimize travel costs and time, possibly skipping non-critical suppliers. In industries with strict time windows for pick-up and delivery (e.g., perishables), optimizing routes within time constraints is crucial. Travel agents can design optimized itineraries where tourists visit a set of sites within a limited time, skipping less attractive sites if necessary. Route optimization for reconnaissance missions or supply drops in the military and defense sector, where real-time intelligence and time constraints may require skipping certain points, is as necessary as route optimization in air and maritime transport, urban waste collection, street sweeping, and snow removal.

The extension of the classical OP to incorporate multiple drones that cooperate with a truck to visit a subset of the input nodes was presented in the paper titled "The Orienteering Problem" [19]. In this case, multiple drones have limited battery endurance. Thus, they can either move together with the truck at no energy cost for the battery or be launched by the truck onto short flights that must start and end at different customer locations.

A practical application of the OP can be found in team orienteering races [4, 25]. In this sport, participants start at a designated control point, aim to visit as many checkpoints as possible and return to the starting point within a set time limit. Each checkpoint has a specific score, and the objective is to maximize the total score collected. In this context, the OP involves selecting a subset of vertices and determining the shortest Hamiltonian path among them so that the OP can be viewed as a combination of the knapsack problem (KP) and TSP. While the TSP focuses on minimizing travel time or distance, the OP aims to maximize the total score with the flexibility of not needing to visit all checkpoints. Finding the shortest path between selected checkpoints is crucial to visiting as many as possible within the available time.

In recent years, the OP, AOP, TSP, and ATSP have been applied in many practical applications, leading to the development of numerous exact algorithms, heuristics, and metaheuristics [21].

The TSP is difficult to solve because it is a combinatorial optimization problem with a solution space that grows factorially with the number of places. Specifically, for n places, there are $(n-1)!$ possible routes, which makes the problem computationally infeasible to solve exactly for even moderately large numbers of places. The TSP is classified as an NP-hard problem, meaning there is no known polynomial-time algorithm that can solve all instances of the TSP exactly. Unlike some problems where a structured or hierarchical approach can be applied, the TSP has no inherent structure that makes it easier to solve. The connections between places (distances) can vary non-uniformly, making it difficult to apply simple rules to reduce the problem space. While heuristic and approximation algorithms (e.g., genetic algorithms, simulated annealing, or the nearest neighbor algorithm) can provide good solutions, finding the exact optimal solution is challenging. These methods do not guarantee the optimal solution, and evaluating the quality of an approximate solution can be difficult. Of course, solving modifications of TSPs such as GTSP, OP, ATSP, etc., also falls into the group of NP-hard problems. However, for practical purposes, it is necessary to find an optimal, or at least good enough, solution. Therefore, many publications have focused on optimizing these problems in the last decade.

The paper titled "Genetic Algorithm with Neighbor Solution Approach for Traveling Salesman Problem" [24] explores an innovative method to solve the TSP using a hybrid approach that combines the strengths of genetic algorithms with a neighbor solution technique to improve the search for an optimal solution. In their approach, the genetic algorithm is used to generate a population of potential solutions, which are then refined through a neighbor solution method. This technique helps in exploring the solution space more effectively by making small adjustments to existing solutions, thereby improving their quality. The results of the study show that this combined approach is effective in finding high-quality solutions for the TSP, offering improvements in both solution accuracy and computational efficiency compared to traditional methods. The paper concludes that the genetic algorithm with the neighbor solution approach is a promising method for solving the TSP and potentially other combinatorial optimization problems.

The paper titled "A Novel Neural Approximation Technique for Integer Formulation of the Asymmetric Travelling Salesman Problem" [19] presents a new approach to solving the ATSP using neural networks. The author proposes a neural approximation technique that formulates the ATSP as an integer programming problem. This approach leverages the capabilities of neural networks to approximate solutions to this complex combinatorial problem. The paper details the design of the neural network model, including its structure, training process, and how it integrates with the integer programming formulation of the ATSP. The neural network generates approximate solutions, which are then refined to meet the integer constraints of the problem. The results demonstrate that this neural approximation technique is effective in providing high-quality solutions to the ATSP, with significant improvements in computational efficiency. This approach shows promise in addressing the challenges posed by asymmetric costs in route optimization, offering a novel method for tackling similar optimization problems in operational research.

A combinatorial optimization problem that involves determining the most valuable route through a set of locations, given specific constraints on distance or time, is explored in the paper "The Orienteering Problem" [8]. The authors discuss various formulations of the problem, including its applications in real-life scenarios such as tourism, logistics, and robotics. They also present algorithms for solving the problem, ranging from exact approaches to heuristics, illustrating the trade-offs between computational efficiency and solution accuracy. An efficient center-of-gravity heuristic is introduced, which outperforms heuristics found in the literature.

The paper titled "New Formulations for the Orienteering Problem" [14] presents new mathematical formulations to solve the OP. The authors propose two novel formulations aimed at enhancing the computational efficiency and accuracy of existing models by integrating different mathematical programming techniques. The first formulation is a mixed-integer linear programming (MILP) model that optimizes the route planning aspect, while the second formulation improves the efficiency of solving larger instances of the problem. The paper also compares the performance of the proposed models with existing approaches using CPLEX 12.5. Computational experiments show that both new formulations outperform existing models, and the proposed formulations are able to solve all benchmark instances that had previously only been solved using specialized heuristics.

The paper titled "The Orienteering Problem with Variable Profits" [10] addresses the OP in the context of varying profits associated with visiting different locations. The problem involves finding the optimal route through a set of locations to maximize total profit while respecting constraints such as time or capacity. In this study, the authors introduce the concept of variable profits, where the profit associated with visiting each location can change based on certain factors. They propose a model and algorithm to solve this extended version of the OP efficiently. The OP with variable profits (OPVP) is formulated on a complete undirected graph with the depot located at vertex 0. Each vertex, except the depot, is associated with a profit value and a selection parameter ranging from 0 to 1. A vehicle can visit a vertex multiple times, gathering a portion of the profit specified by the selection parameter during each visit. Alternatively, another model allows the vehicle to spend continuous time at each vertex, collecting a percentage of the profit based on the time spent at that location. The objective is to find the route that maximizes profit for the vehicle, starting and ending at the depot, while adhering to the travel time constraint. The paper contributes to the field of operations research by exploring a more realistic scenario where profits are variable rather than fixed, and it provides a framework for solving such problems effectively.

An enhanced branch-and-cut algorithm for solving large-scale orienteering problems, achieving significant improvements in solving benchmark instances, was proposed in the paper titled "A Revisited Branch-and-Cut Algorithm for Large-Scale Orienteering Problems" [16]. The authors presented a revisited version of the branch-and-cut algorithm for the OP that brings multiple contributions together. They developed two heuristics for cycle problems, building on existing TSP methods. An efficient variable pricing procedure was designed for the OP, reducing repetitive calculations.

The application of artificial neural networks (ANNs) to address the OP is presented in the paper titled "Using Artificial Neural Networks to Solve the Orienteering Problem" [23]. The authors develop an ANN-based approach to solve the OP, leveraging neural networks' learning and pattern recognition capabilities to identify optimal or near-optimal solutions. The paper details the design and implementation of the neural network model, including the architecture, training process, and evaluation methodology. The authors compare the performance of their ANN-based solution with traditional optimization methods, demonstrating that the neural network approach can effectively solve the OP, particularly for instances where other methods struggle. The results indicate that ANNs provide a promising alternative for solving the OP, offering competitive solution quality and computational efficiency. This research also highlights the potential of neural networks in solving complex optimization problems beyond traditional techniques.

The team orienteering problem (TOP) is described in the paper titled "A Branch-and-Cut Algorithm for the Team Orienteering Problem" [2]. The TOP involves a team of individuals collectively selecting routes to maximize the total profit while visiting a set of locations within a specified time limit. In this study, the authors propose a branch-and-cut algorithm to solve the TOP efficiently. They introduced a new two-index formulation that features a polynomial number of variables and constraints. This streamlined formulation, enhanced by the inclusion of continuity constraints, has been tackled using the branch-and-cut algorithm. This algorithm combines techniques from branch-and-bound methods with cutting planes to improve the solution quality and computational performance. The authors successfully resolved 327 out of 387 reference instances to optimality, surpassing previous methodologies by 26 instances. Additionally, 24 previously unresolved instances have now been closed to optimality.

3. Mathematical Formulation of the AOP Model

A complete edge-valued ordinary digraph defined by the set of nodes V and the set of oriented edges Y is given. Let the nodes $v_1, v_2 \ldots, v_n$ in the set of nodes $V = \{v_1, v_2 \ldots, v_{n+1}\}$ represent the locations of possible visits and let a score of $c_i > 0$, where $i \in V\,\{n+1\}$, be defined for each of these nodes. Let the node v_{n+1} represents the place in which the route starts and ends. Let the oriented edge $[v_i, v_j] \in Y$, where $i = 1, \ldots, n+1$, $j = 1, \ldots, n+1$ and $i \neq j$ represents the minimum path from node $i = 1, \ldots, n+1$ to node $j = 1, \ldots, n+1$ and its evaluation $t_{i,j}$ represents the traversal time from node $i = 1, \ldots, n+1$ to node $j = 1, \ldots, n+1$. Let the time limit (maximum route duration) $T_{\max}$ be further defined.

The task is to decide which nodes should be visited within the planned route and the order of visiting these nodes so that the maximum total score (point gain) is achieved within a defined time limit.

Notation:

n	number of nodes generating a point gain (score) in the transport network,
c_j	value of the point gain obtained by visiting node $j = 1, \ldots, n$ (for node $n+1$ the value $c_{n+1} = 0$),
$t_{i,j}$	travel time from node $i = 1, \ldots, n+1$ to node $j = 1, \ldots, n+1$, where $i \neq j$,
$T_{\max}$	time limit given for node visits.

In order to model the required decisions, we introduce the following variables into the mathematical model:

$x_{i,j}$ a binary variable representing the traversal from the node $i = 1, \ldots, n+1$ to $j = 1, \ldots, n+1$ and $i \neq j$; if $x_{i,j} = 1$ after the optimization is completed, then the traversal from node $i = 1, \ldots, n+1$ to node $j = 1, \ldots, n+1$ will occur; if $x_{i,j} = 0$, the traversal will not occur,

z_i a non-negative variable ensuring that an undesired type of subcycle (one that does not pass through the node where the route starts and ends) does not occur, for $i = 1, \ldots, n$.

Mathematical model of the optimization problem is as follows:

$$\max(x, z) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c_j x_{i,j}, \tag{1}$$

Subject to:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_{i,j} \leq 1 \qquad \text{for } i = 1, \ldots, n+1, \tag{2}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n+1} x_{i,j} = \sum_{\substack{i=1 \\ i \neq j}}^{n+1} x_{j,i} \qquad \text{for } j = 1, \ldots, n+1, \tag{3}$$

$$\sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ j \neq i}}^{n+1} t_{i,j} x_{i,j} \leq T_{\max}, \tag{4}$$

$$z_i - z_j + n x_{i,j} \leq n - 1 \qquad \text{for } i = 1, \ldots, n, j = 1, \ldots, n \text{ and } i \neq j, \tag{5}$$

$$x_{i,j} \in \{0, 1\} \qquad \text{for } i = 1, \ldots, n+1; \; j = 1, \ldots, n+1 \text{ and } i \neq j, \tag{6}$$

$$z_i \in R_0^+ \qquad \text{for } i = 1, \ldots, n. \tag{7}$$

Function in Eq. (1) represents the optimization criterion – the total score (points gain) achieved within the time limit $T_{\max}$. The constraints in group (2) ensure that a traversal to just one other node will be made from each visited node $i = 1$ and node $n + 1$, where the route starts. The constraints in group (3) guarantee the continuity of the path at the visited nodes. Constraint (4) guarantees that a defined time limit is respected in the route planning. Group of constraints (5) ensures that undesired subcycles (subcycles not passing through the node where the route starts and ends) do not occur during route construction. Finally, groups (6) and (7) define the domains of the variables used in the model.

4. AOP Model with Profitable Penalty

Now, we extend the formulation of the problem presented in Section 3 by adding another aspect – the penalty for exceeding the time limit $T_{\max}$. We rename the original time limit (maximum route duration) $T_{\max}$ as the baseline time limit (maximum route duration without penalty). Next, we define a set of possibilities to increase it M, where $|M| = m$, and let be defined time values τ_i, where $i = 1, \ldots, m$, representing the values by which the value of the basic time limit is increased. Increasing the base time limit $T_{\max}$ can results in extra point gains from node visits made in the time interval $(T_{\max}; T_{\max} + \sum_{i=1}^{m} \tau_i\rangle$, but it is also a source of penalty points that decreases the total score, whereby increasing the time limit by the value, τ_i, for $i = 1, \ldots, m$, decreases the value of the total score by the value $\bar{c_i} > 0$.

The task is to decide on the nodes to be visited within the planned route and the order of visiting these nodes so that the maximum total score is achieved within a defined time limit.

Notation

n	number of nodes generating a point gain (score) in the transport network,
m	number of possibilities for increasing the basic time limit,
c_j	value of the point gain obtained by visiting node $j = 1, \ldots, n$ (for the node $n+1$ the value $c_{n+1} = 0$),
$\bar{c_j}$	value of the penalty resulting from visits to nodes when the time limit is increased by $j = 1, \ldots, m$,
$t_{i,j}$	travel time from node $i = 1, \ldots, n+1$ to node $j = 1, \ldots, n+1$, where $i \neq j$,
$T_{\max}$	basic time limit given for node visits,
τ_i	the value of the time step $i = 1, \ldots, m$ allowing the time limit increase.

In order to model the required decisions, we introduce the following variables into the mathematical model:

$x_{i,j}$	a binary variable representing the order of visits to nodes $i = 1, \ldots, n+1, j = 1, \ldots, n+1$ and $i \neq j$; if $x_{i,j} = 1$ after the optimization is completed, then the traversal from node $i = 1, \ldots, n+1$ to node $j = 1, \ldots, n+1$ will occur; if $x_{i,j} = 0$, the traversal will not occur;
y_i	a binary variable representing the existence of possibilities to increase the basic time limit by the time interval $\sum_{j=1}^{i} \tau_j$, where $i = 1, \ldots, m$; if $y_i = 1$ after the optimization is completed, then the value of the base limit will increase by the value $\sum_{j=1}^{i} \tau_j$; if $y_i = 0$, the increase will not occur;
z_i	a non-negative variable ensuring that an undesired type of subcycle (one that does not pass through the node where the route starts and ends) does not occur, for $i = 1, \ldots, n$.

Mathematical model of the optimization problem is as follows:

$$\max(x, y, z) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c_j x_{i,j} - \sum_{j=1}^{m} \bar{c_j} y_j. \tag{8}$$

Subject to constraints (2), (3), (5)–(7) supplemented by constraints:

$$\sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ j \neq i}}^{n+1} t_{i,j} x_{i,j} \leq T_{\max} + \sum_{k=1}^{m} y_k \sum_{p=1}^{k} \tau_p, \tag{9}$$

$$\sum_{k=1}^{m} y_k \leq 1, \tag{10}$$

$$y_i \in \{0; 1\} \qquad \text{for } i = 1, \ldots, m. \tag{11}$$

Function in Eq. (8) represents the optimization criterion – the total score achieved within one of the permissible time limits. Constraint (9) ensures that the basic time limit is respected when planning the route and allows extending it. Constraint (10) ensures that the basic time limit is extended by at most one of the possible time intervals. The group of constraints (11) defines the domains of the variables used in the model.

5. Case Study

Mathematical models presented in Chapters 3 and 4 will be applied and validated in the SKI ADVENTURE sports discipline in the Czech Republic. This is a ski orienteering race. The race takes place on the Czech side of the CHKO Jizerské hory (Jizera Mountains Protected Landscape Area), in the area of the Jizerská magistrála (Jizera cross-country route) and its surroundings, on a plateau at an altitude of around 1000 meters above sea level.

A similar version of this race is the summer orienteering sport on mountain bikes (MTBO).

5.1 The Race Rules

The race rules differ from the classical ski orienteering primarily in that the participants of the race form racing pairs (hereinafter referred to as teams) in 7 categories (MM – male/male, MM+ – male/male with the sum of their ages over 90 years, MD – male/female, MD+ – male/female with the sum of their ages over 90 years, DD – female/female, DD+ – female/female with the sum of their ages over 90 years, RD – parent/child under the age of 15). During the race, the distance between the pair must be minimal, and they must be allowed to communicate without technical devices. The start and finish of the race are located at the same point (K33), and there are 32 checkpoints (K1–K32) along the route. These checkpoints are positioned such that teams cannot visit all of them during the race. Passing through the checkpoints K1–K32 a team gains a point score of 10, 20, 30, 40, 50, or 100 points (each checkpoint has only one valid score). The objective for teams is to plan routes that maximize point gain within the 5-hour time limit. A map of the area with the checkpoints marked is given to the teams 12 hours before the race, allowing them to prepare their route strategy.

In order to increase the total point score, there is the option to exceed the 5-hour race time limit. However, after exceeding this limit, teams must expect a penalty in the form of a reduction in the total point score, depending on the amount of time over the limit. The total point score is reduced for each started minute according to the rules (see Tab. I).

	Race completion interval [h:min]	$\bar{c}_j$ [points per minute started]
Penalty 0	$\langle 0:00; 5:00 \rangle$	0
Penalty 1	$(5:00; 5:05\rangle$	2
Penalty 2	$(5:05; 6:00\rangle$	5

Tab. I *Penalty points for every minute started in the race completion interval when exceeding the time limit.*

The maximum time limit for completing the race is 6 hours. However, a team may score more points overall if the point gain minus the penalty is profitable, see Tab. II. Tab. II considers the penalty's profitability for different point values per visited checkpoint beyond the basic time limit. However, a team may visit multiple checkpoints between the time of 5:00:01 and 6:00:00, which is beyond the basic 5-hour time limit, thus increasing the profitability of the penalty. For this reason, in constraint (9) of the mathematical model, the value of τ_i will be increased up to 60 minutes.

5.2 Input Data

The complete, strongly continuous edge-weighted digraph is composed of nodes representing the start/finish of the race and the positions of the individual checkpoints and oriented edges whose ratings represent the traversal times of the ski teams between each pair of nodes in seconds, depending on the performance of the virtual teams (V1, V2, V3), i.e., the teams entering the computational experiments.

The traversal time between each pair of nodes is calculated based on the kilometer distance, the speed of the virtual teams and the terrain elevation. It is true that the more efficient the virtual team, the higher its average speed, see Tab. III. Time distance matrices according to the performance of virtual teams: The fragments of the time requirements of the transitions between nodes are given according to the performance of the virtual teams in seconds, ranging from less to more efficient (time distance matrix **A**, matrix **B**, matrix **C**), see Tab. IV–VI. To each element of the matrix, a constant value of 15 is added, which is needed to orient the virtual teams at the checkpoints K1–K32 and for the virtual teams to decide the next race direction.

5.3 Results of Computational Experiments

The outputs of the optimization are the values of the optimization criteria, the values of the total time spent by the virtual teams on the race route (TT), the values of the total score without penalty, the total value of penalties, the computational complexity (T) and the gap value, which can be calculated using Eq. (12) according to [13].

Race completion time interval [h:min]	$\bar{c}_j$ [points per minute started]	Profitable penalty for different point values per visited checkpoint beyond the basic time limit [points/checkpoint]				
$\langle 0:00; 5:00\rangle$	0	20	30	40	50	100
$\langle 5:00; 5:01\rangle$	2	18	28	38	48	98
$\langle 5:01; 5:02\rangle$	4	16	26	36	46	96
$\langle 5:02; 5:03\rangle$	6	14	24	34	44	94
$\langle 5:03; 5:04\rangle$	8	12	22	32	42	92
$\langle 5:04; 5:05\rangle$	10	10	20	30	40	90
$\langle 5:05; 5:06\rangle$	15	5	15	25	35	85
$\langle 5:06; 5:07\rangle$	20	0	10	20	30	80
$\langle 5:07; 5:08\rangle$	25	–	5	15	25	75
$\langle 5:08; 5:09\rangle$	30	–	–	10	20	70
$\langle 5:09; 5:10\rangle$	35	–	–	5	15	65
$\langle 5:10; 5:11\rangle$	40	–	–	–	10	60
$\langle 5:11; 5:12\rangle$	45	–	–	–	5	55
$\langle 5:12; 5:13\rangle$	50	–	–	–	–	50
$\langle 5:13; 5:14\rangle$	55	–	–	–	–	45
$\langle 5:14; 5:15\rangle$	60	–	–	–	–	40
$\langle 5:15; 5:16\rangle$	65	–	–	–	–	35
$\langle 5:16; 5:17\rangle$	70	–	–	–	–	30
$\langle 5:17; 5:18\rangle$	75	–	–	–	–	25
$\langle 5:18; 5:19\rangle$	80	–	–	–	–	20
$\langle 5:19; 5:20\rangle$	85	–	–	–	–	15
$\langle 5:20; 5:21\rangle$	90	–	–	–	–	10
$\langle 5:21; 5:22\rangle$	95	–	–	–	–	5
$\langle 5:22; 5:23\rangle$	100	–	–	–	–	–

Tab. II *Number of points for the profitability of the penalty based on the score for a given control point.*

	Team V1 average speeds [km.h^{-1}]	Team V2 average speeds [km.h^{-1}]	Team V3 average speeds [km.h^{-1}]
Steep descent	19.5	20	20.5
Gentle descent	16.0	17	18.0
Flat terrain	12.5	14	15.5
Gentle ascent	10.0	11	12.0
Steep ascent	7.5	8	8.5

Tab. III *Table of average speeds in relation to terrain type depending on the performance of the virtual racing pair.*

	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10
K1	0	636	996	759	939	1515	2422	2976	3476	4277
K2	791	0	749	766	1133	1709	2616	3170	3670	4443
K3	1151	749	0	1138	1505	2081	2989	3543	4043	4816
K4	697	685	1138	0	382	958	1865	2420	2919	3692
K5	877	1052	1505	382	0	591	1498	2297	2769	3356
K6	1453	1628	2081	958	591	0	1095	1865	2452	3264
K7	2360	2535	2989	1865	1498	1095	0	1045	2200	3021
K8	2645	2819	3543	2150	2122	1776	761	0	1289	2139
K9	3320	3494	4043	2825	2600	2283	1941	1195	0	1770
K10	3797	4065	4518	3395	3181	2994	2972	2122	1601	0

Tab. IV *Fragment of time distance matrix A for team V1.*

	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10
K1	0	586	908	700	861	1375	2185	2692	3135	3861
K2	719	0	671	692	1020	1534	2344	2851	3294	4020
K3	1040	671	0	1018	1346	1860	2670	3177	3620	4346
K4	631	623	1018	0	343	857	1667	2174	2617	3343
K5	791	951	1346	343	0	529	1339	2060	2481	3114
K6	1306	1465	1860	857	529	0	979	1675	2198	2947
K7	2116	2275	2670	1667	1339	979	0	947	1973	2709
K8	2391	2551	2946	1943	1910	1602	704	0	1157	1919
K9	2985	3144	3539	2536	2337	2054	1765	1076	0	1605
K10	3564	3723	4118	3115	2877	2710	2674	1910	1460	0

Tab. V *Fragment of time distance matric B for team V2.*

	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10
K1	0	544	835	636	781	1246	1978	2444	2843	3535
K2	659	0	607	632	928	1392	2124	2590	2989	3690
K3	950	607	0	921	1217	1681	2413	2880	3278	3979
K4	576	572	921	0	311	776	1507	1974	2372	3073
K5	721	868	1217	311	0	480	1211	1868	2248	2878
K6	1186	1332	1681	776	480	0	886	1520	1993	2684
K7	1918	2064	2413	1507	1211	886	0	866	1788	2456
K8	2184	2330	2820	1774	1738	1460	656	0	1050	1740
K9	2713	2859	3278	2302	2123	1868	1620	980	0	1469
K10	3371	3527	3876	2970	2630	2479	2432	1738	1344	0

Tab. VI *Fragment of time distance matrix C for team V3.*

$$gap = \frac{Best\,Bound - Best\,Solution}{Best\,Solution} \cdot 100, \tag{12}$$

where *BestSolution* (BS) is the value of the best integer solution found so far *BestBound* (BB) is the value of the upper estimate of the best integer solution found so far.

The computational experiments were performed on a PC equipped with the Intel(R) Core(TM) i9-10900X processor with the parameters: 3.70 GHz and 64 GB RAM with an installed version of the optimization software – FICO Xpress IVE Version 9.0 (64-bit, release 2022).

The optimization runs for virtual team V1 without penalty and with a profitable penalty are documented in graphs Fig. 1 and Fig. 2. The graphs include the time evolution of the lower and upper estimates of the values of the optimization criteria and the time evolution of the gap value. The optimization runs for virtual teams V2 and V3 without penalization and with cost-effective penalization followed similar trends. In a relatively short time after the start of the optimization computation, the gap value dropped sharply to the near-optimal solution, and in the remaining computation time, the gap value dropped significantly slower and for different lengths of time. This is documented by the different values of the computational demands T (Tab. VII).

	TT [h:min:s]	$\sum_{j=1}^{n+1} c_j$ [points]	$\sum_{j=1}^{m} \bar{c}_j$ [points]	$\sum_{j=1}^{n+1} c_j - \sum_{j=1}^{m} \bar{c}_j$ [points]	T [s]	gap [%]
BS V1 without penalty	04:59:44	640	0	640	1100.2	0
BS V1 with profitable penalty	04:58:55	640	0	640	1401.1	$2.7 \cdot 10^{-12}$
BS V2 without penalty	04:58:24	700	0	700	2806.8	$7.5 \cdot 10^{-12}$
BS V2 with profitable penalty	05:02:42	710	6	704	14715.8	0
BS V3 without penalty	04:59:46	760	0	760	7302.4	0
BS V3 with profitable penalty	05:04:34	770	10	760	8208.5	0

Tab. VII *Overview of the results of optimizations depending on the performance of virtual teams and the possibility of profitable penalties.*

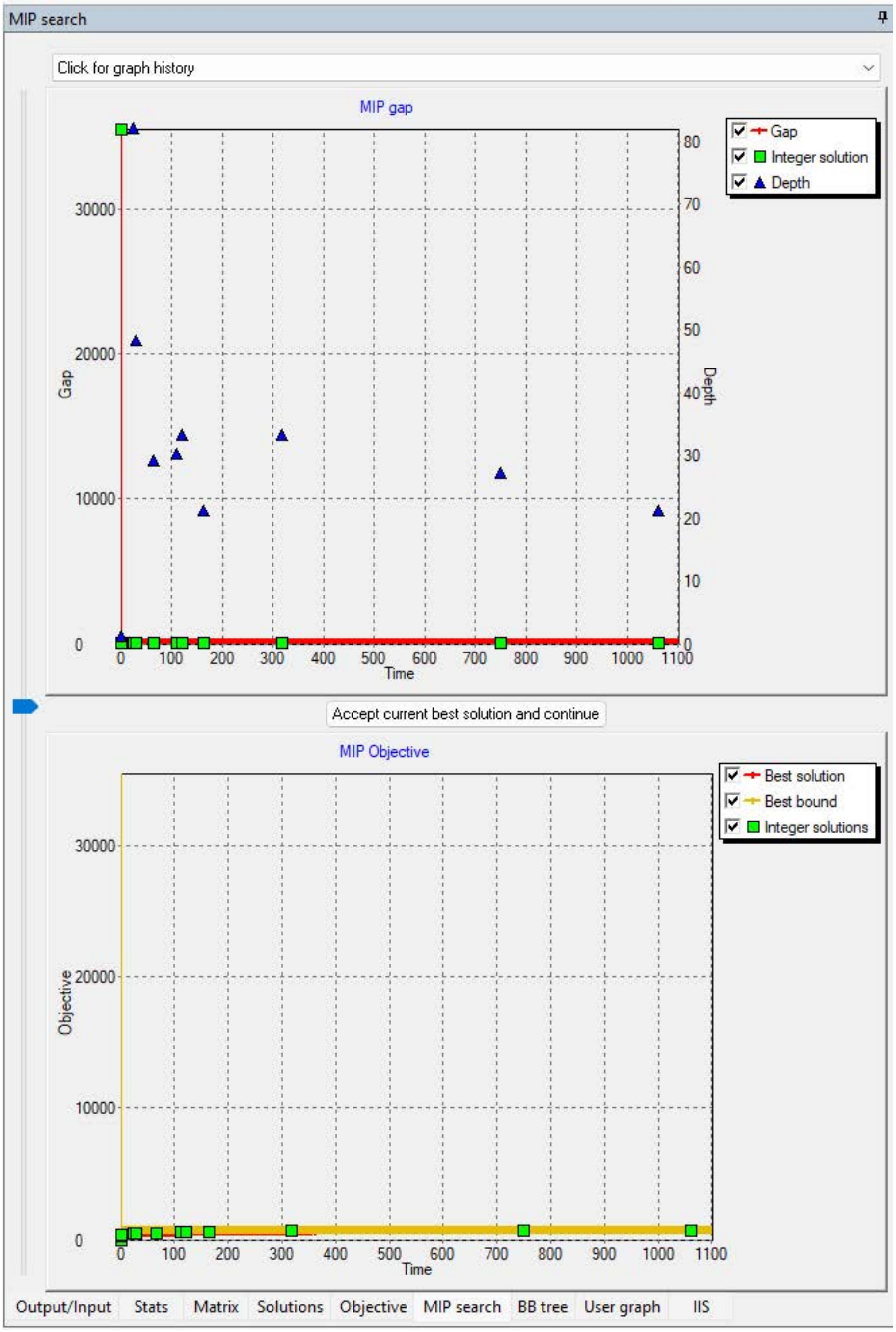

Fig. 1 *Graph of the computation process and a gap of BS V1 without penalty.*

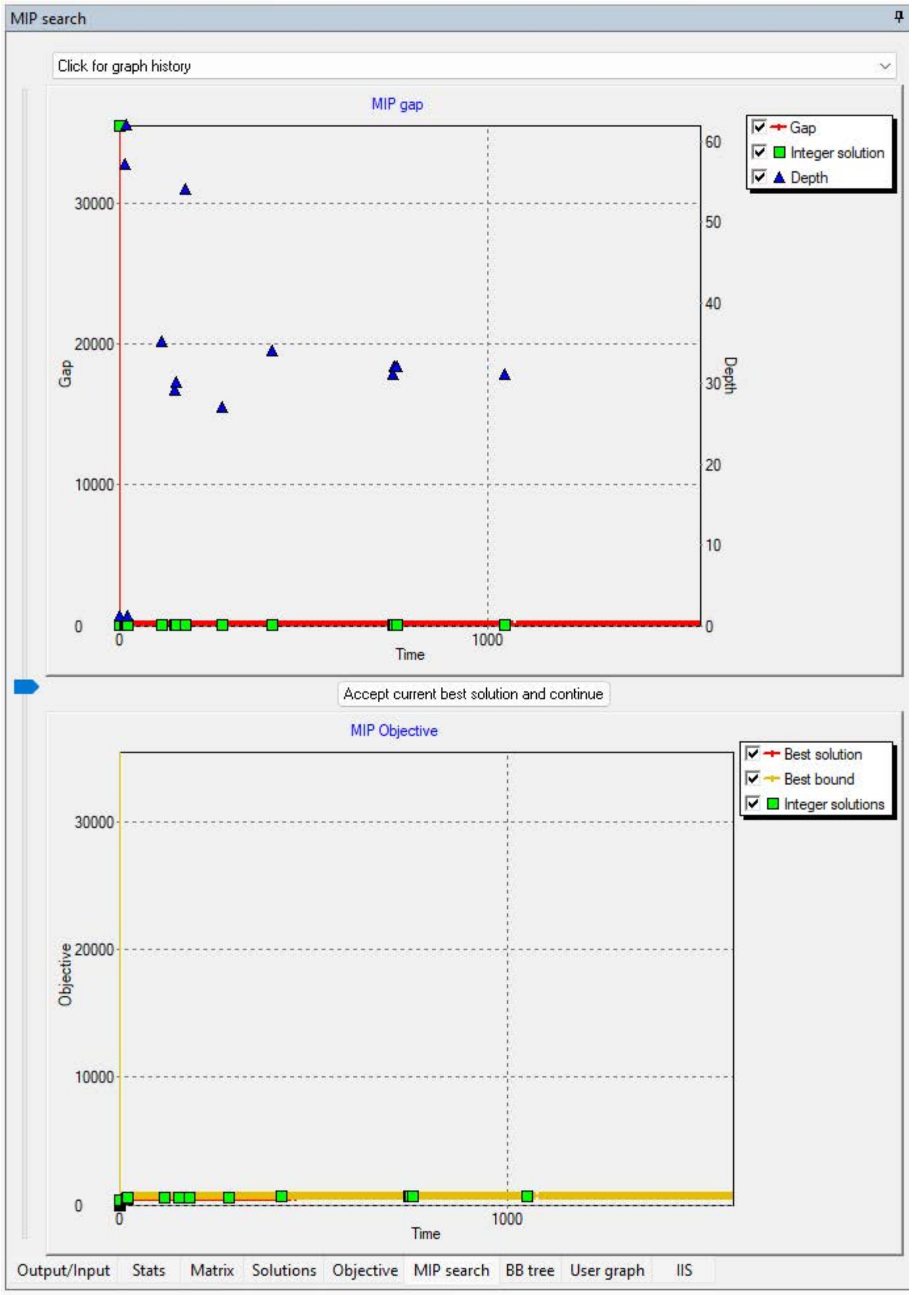

Fig. 2 *Graph of the computation process and a gap of BS V1 with profitability penalty.*

5.4 Comparison of Computational Experiments with Real Race Results

To compare the results of the optimizations, the final list of real results from the top five teams (team R1, team R2, team R3, team R4, team R5) in the MM – male/male category from 2020 will be used. The results will be arranged in descending order based on the point scores. In the case of identical point scores, the results will be arranged by the total time spent on the race course. The ranking will also include the results of virtual teams based on performance from computational experiments (Tab. VIII).

When we compare the results of the real teams with the virtual teams, in cases without penalties, the most efficient real team loses 120 points to team V3 and 60 points to team V2. In cases allowing for profitable penalties, the most efficient real team loses 120 points to team V3 and 60 points to team V2. However, in both scenarios, the most efficient real team always achieves better results than team V1.

	TT [h:min:s]	$\sum_{j=1}^{n+1} c_j$ [points]	$\sum_{j=1}^{m} \bar{c}_j$ [points]	$\sum_{j=1}^{n+1} c_j - \sum_{j=1}^{m} \bar{c}_j$ [points]
BS V3 without penalty	04:59:46	760	0	760
BS V3 with profitability penalty	05:04:34	770	10	760
BS V2 with profitability penalty	05:02:42	710	6	704
BS V2 without penalty	04:58:24	700	0	700
R1	04:57:43	640	0	640
BS V1 with profitability penalty	04:58:55	640	0	640
R2	04:59:18	640	0	640
BS V1 without penalty	04:59:44	640	0	640
R3	05:02:35	640	6	634
R4	04:55:58	600	0	600
R5	04:51:21	590	0	590

Tab. VIII *Comparison of results of computational experiments with real race results in MM category.*

6. Discussion and Conclusions

The AOP is an optimization problem that finds applications in a wide range of areas. The main contribution of this paper is the extension of the existing spectrum of exact approaches for solving the AOP by introducing mathematical models that allow exceeding of the time limit in order to gain a benefit, such as additional points, even in cases where exceeding the time limit results in a penalty. In the proposed mathematical model for solving the AOP with a profitable penalty, the penalty for each time unit exceeded can be either constant or variable. If the total score becomes worse due to the penalty, the option to exceed the time limit will not be used. In such cases, the penalty will be assessed as unprofitable.

The case study demonstrates that the organization of the Ski Adventure race is one of the applications of the proposed mathematical model, with the option to exceed the time limit and apply variable penalties. The proposed mathematical model can be used by sports team managers or the teams themselves, as well as by the race organizers. In preparing their race strategy, teams receive a map of the area with the locations of the checkpoints and the point values of each checkpoint 12 hours before the start of the race. The computational times achieved during the computational experiments show that these times are significantly shorter than this 12-hour window. This fact means the teams have enough time to prepare their race strategy to maximize their point score. Based on the mathematical model results, race organizers can better predict the strategies of different teams and, based on these predictions, appropriately place refreshment stations for competitors, medical teams, or other types of service stations.

From the computational experiments with the proposed models it was also found that:

- The computational times varied significantly across different experiments, ranging from 1100.2 seconds (for team V1 and the variant without penalty) to 14715.8 seconds (for team V2 with profitable penalty),
- The experiments without penalties (for all teams V1–V3) were less time-consuming,
- The optimal solution was achieved at different stages of the optimization process. The fastest convergence to the optimal solution occurred in the computational experiment for team V2 with a profitable penalty (9.8% of the computational time), while the slowest was for team V1 without penalties (99% of the computational time). This fact shows different convergence speeds to the optimal solution. It is impossible to predict the computational time in an exact solution based on the branch-and-bound method.

Further applications of the mathematical model with profitable penalties can be found in other sports, distribution logistics, or tourism. In distribution logistics, the proposed model can be used to assess the efficiency of serving additional customers when exceeding a time limit, such as the length of a driver's shift. In tourism, the model can be applied to assess the effectiveness of visiting other landmarks within a tourist trip of a predefined length.

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