EQUATIONS OF MOTION OF MECHANICAL SYSTEMS WITH SWITCHABLE CONSTRAINTS

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Abstract: The article deals with a numerical and analytical approach to solving the equations of motion, which enables to treat the considered problems with the change of system structure or number of degrees of freedom without interrupting the numerical integration process. The described methodology allows effectively incorporate switchable constraints in the systems in accordance with their flexible structures. The crucial idea is based on the formulation of the resulting differential-algebraic equations into a saddle point system, where the switchable constraints are represented by a sign matrix with variable rank. In connection with this property, a pseudoinversion is applied to eliminate algebraic variables and transform the problem to the first order system of ordinary differential equations. Moreover, the time independent case leads to linear autonomous systems with non-diagonalizable matrices, as is proved. The relevant numerical scheme is based on Runge-Kutta methods, that correspond to the power series of the resulting matrix exponential for time independent problems. The methodology presented is illustrated on the idealized two-mass oscillator with a switchable constraint. The numerical experiments performed range from initial stages, through simple transient cases to damped intentional control. The advanced applications can be found in robotics, active and controlled systems, and in the simulations of complex systems in biology and related areas. Moreover, the methodology can also be applied in the simulation of transport systems, especially in relation to vehicle technology, a quarter car suspension system, a vibration control mechanism, a torsion system with a clutch, and machine balancing and storage should to be highlighted.

Key words: *differential-algebraic equations, saddle point system, switchable constraint, pseudoinversion, Runge-Kutta method, two-mass oscillator, intentional control*

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1. Introduction

Current technology requires simulation capabilities to fluently calculate systems with inherent or intentional nonlinearities, for example, static and dynamic friction, changed structure, or nonlinear control [24]. Therefore, correct simulation of nonlinearities type friction with the change of the number of degrees of freedom, with the continuous integration process, is generally important in a number of engineering applications, especially in robotics, vehicles, and mechanical systems. Autonomous transportation systems suppose flexible connection of transportation means. Hybrid cars have complex traction systems with flexible roles of driven units. The demand for the correct design of digital twins amplifies the requirement for flexible calculation approaches.

Parallel to the above, another feature of the simulation approach should be the possibility of solving the singular cases of the system. More precisely, cases where the system itself decides on the continuous internal dynamics — the way is just to let run the integration with the locked joints responsible for the singular position and, after sufficient time steps, to unlock them again in a new and regular position. The applications can be found in various fields, including robotics, active and controlled systems, as well as in the simulations of complex biological systems and related areas [24]. The simulation of the transport systems can utilize the presented methodology as well. Concerning the vehicle technology emphasize at least a quarter car suspension system [21], a vibration devourer, a torsion system with a clutch and machine balancing and storage, see [14] and from a wider perspective [9].

From the theoretical point of view, we deal with a constrained mechanical system, defined by ordinary differential equations (ODEs) subjected to additional algebraic equations. In other words, the governing equations of a such mechanical system form a system of differential-algebraic equations (DAEs). Since DAE involves both differentiation and integration processes, which may additionally be intertwined, DAE is more challenging to solve than ODE, as presented in [2]. The complexity is proportional to an index of DAEs, see [6]. A widely used approach consists of a reformulation of DAEs into an equivalent system of ODEs via the elimination of algebraic variables (i.e., the associated Lagrange multipliers) under an appropriate choice of generalized coordinates, for a comprehensive overview of classical and contemporary techniques, see [3, 17] and references cited therein. From the recently published literature, mention at least a paper [18], where model reduction is applied.

In this work, we follow the theory of DAE systems and develop a methodology incorporating switchable linear constraints, represented by a $(0, \pm 1)$-matrix with variable rank, thereby enabling rigorous analysis of the resulting saddle point systems. We also provide a numerical simulation in the MATLAB platform, arising from this methodology, focused on the simplest benchmark. The core of this paper consists of three sections. We introduce the methodological concept in Section 2 and present the approach on a two-mass oscillator with a switchable constraint in detail in Section 3. Then, the attention is paid to numerical simulations in Section 4. Finally, we give a brief conclusion.

2. Methodology

The advantage of powerful processors and large available memory enable to create of an efficient mathematical model of the mechanical system on the basis of the equation of motion formalism, which includes the possibility of including the flexible structure of the described system.

The mechanical system studied is assembled of n rigid bodies, localized by generalized position coordinates $\mathbf{s} = (s_1, \ldots, s_n)^{\mathrm{T}}$. In order to avoid ambiguity, the one dimensional spatial scenario is considered, i.e., a position of the jth mass element at time $t > 0$ coincides with the function $s_j(t)$, $j = 1, \ldots, n$. This specification does not lose any generality and makes the description as clear as possible.

In line with classical mechanics [1], the movement of these rigid bodies is defined by kinematics, subjected to m holonomic constraints $g_k(t, \mathbf{s}) = 0$, $k = 1, \ldots, m$. We assume that $m < n$ and the constraints form a non-redundant set. Derivation of the resulting equations of motion is based on the well-known Lagrangean principle [8] and leads to the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{K}}{\partial \dot{s}_j} - \frac{\partial \mathcal{K}}{\partial s_j} = \mathcal{Q}_j + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial s_j}, \qquad j = 1, \ldots, n, \tag{1}$$

where $\dot{\mathbf{s}} = (\dot{s}_1, \ldots, \dot{s}_n)^{\mathrm{T}}$ are the generalized velocities, $\mathcal{K} = \mathcal{K}(t, \mathbf{s}, \dot{\mathbf{s}})$ denotes kinetic energy, $\mathcal{Q}_j = \mathcal{Q}_j(t, \mathbf{s}, \dot{\mathbf{s}})$, $j = 1, \ldots, n$ are general forces, and $\lambda_k = \lambda_k(\mathbf{s}, \dot{\mathbf{s}})$, $k = 1, \ldots, m$, Lagrange multiplier functions. More precisely, quantity $\mathcal{Q}_j$ can be decomposed into

$$\mathcal{Q}_j = \overline{\mathcal{Q}}_j - \frac{\partial \mathcal{V}}{\partial s_j} - \frac{\partial \mathcal{D}}{\partial \dot{s}_j}, \tag{2}$$

where $\mathcal{V} = \mathcal{V}(t, \mathbf{s}, \dot{\mathbf{s}})$ is potential energy, $\mathcal{D} = \mathcal{D}(t, \mathbf{s}, \dot{\mathbf{s}})$ dissipative function and $\overline{\mathcal{Q}}_j = \overline{\mathcal{Q}}_j(t, \mathbf{s}, \dot{\mathbf{s}})$ remaining external forces, respectively.

In the standard design of the equations of motion, the Lagrange multipliers are the linear combination of the internal forces and do not have any direct physical meaning. In contrast, within this approach, the constraints identical to the selected coordinates produce the internal forces acting in the kinematic joint or constraints directly. Precisely, the advantageous definition of constraints is crucial for the functionality of the algorithm.

In other words, since all members on the right-hand side of (1) are forces, the special case, when $\left|\frac{\partial g_k}{\partial s_j}\right| = 1$ for some $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, implies that the Lagrange multipliers are directly forces acting along selected coordinates. To achieve this effect, suitable coordinates are used to describe the system. Then, each kinematic joint under interest should be described in the meaning of independent coordinates of the system. In the limit case, all kinematic joints are disconnected and the Lagrange equations can be derived together with constraint equations. After performing all the steps of the Lagrange algorithm, system (1)

with (2) can be rewritten in the matrix form as follows (see [25]):

$$\left[\begin{array}{cccc|cccc} m_{11} & m_{12} & \cdots & m_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\ m_{21} & m_{22} & \cdots & m_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} & g_{1n} & g_{2n} & \cdots & g_{mn} \\ \hline g_{11} & g_{12} & \cdots & g_{1n} & 0 & 0 & \cdots & 0 \\ g_{21} & g_{22} & \cdots & g_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & g_{12} & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} & 0 & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \ddot{s}_1 \\ \ddot{s}_2 \\ \vdots \\ \ddot{s}_n \\ \hline -\lambda_1 \\ -\lambda_2 \\ \vdots \\ -\lambda_m \end{array}\right] = \left[\begin{array}{c} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_n \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right], \quad (3)$$

where

$$m_{ij} = \frac{\partial^2 \mathcal{K}}{\partial \dot{s}_j \partial \dot{s}_i}, \qquad g_{kj} = \frac{\partial g_k}{\partial s_j} \quad i, j \in \{1, \ldots, n\},\ k \in \{1, \ldots, m\}.$$

The size of the differential-algebraic problem (3) (i.e., number of equations) grows rapidly with the complexity of the mechanical system, and the matrix form produces block structure, see [7]. Solving the problem (3) requires proper matrix organization and allows a very flexible approach, see [20].

Under this framework, $g_{kj} \in \{-1, 0, 1\}$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$, where the value 0 means no link between coordinates (i.e., they are free and independent) and the value ± 1 creates the constraint (i.e., related coordinates are fixed together and firmly connected). The key idea of the methodology presented is that it is possible to switch between these zero and non-zero values during the simulation process without interrupting it. As a result, we are able to incorporate a switchable constraint in this way and treat it as the rheonomic one in general. On the other hand, the fixed (active) constraint, i.e., $|g_{kj}| = 1$ for at least one $j \in \{1, \ldots, n\}$, can be considered scleronomic. In practice, representative examples are the situations when the number of degrees of freedom changes. It is a loss of mobility (i.e., ankylosis) due to the friction, and intervention of the active element, but even simple situations like the engagement/disengagement of the vehicle clutch, freewheel lock/unlock, etc., see [23].

The switching on and off of constraints introduces strong nonlinearities into the whole system and thus it makes the simulation process complex and expensive. While switching off ($1 \to 0$ or $-1 \to 0$) can be done anytime on the basis of external requirement (i.e., control input) or on the basis of the defined condition (i.e., external forces exceed the static friction and the joint starts the relative movement), switching on ($0 \to 1$ or $0 \to -1$) means the connection of bodies or the lock of the kinematic joint and it can only be done under certain circumstances, e.g., when the relative velocity of the monitored kinematic joint is zero. Therefore, it is necessary to identify such proper time instants in a passive way or apply a control strategy to achieve the required kinematic condition.

The advantage of the methodology presented is that these steps do not influence the integration of equations of motion in any case and the simulation (meaning numerical) process can continue in spite of the structural changes of the mechanism. This can be considered the main benefit of this approach, since is not necessary to

stop the integration process due to the change in the number of degrees of freedom during movement (done by friction, active system, singular case, etc.). In contrast, stopping integration leads to a new definition of initial conditions, shortening of the integration step, etc. The consequence is a loss of continuity, more extended time of simulation process, and inconsistent results in the very beginning phase of simulation. These phenomena are eliminated by the presented approach, which is stable, reliable, and time-effective.

2.1 Non-switchable Constraints

As the first scenario assume that constraints are non-switchable, from the mathematical point of view we study the second-order ODE system subjected to m linear constraints

$$g_k(\mathbf{s}) = g_{k1}s_1 + \cdots + g_{kn}s_n = d_k, \qquad k = 1, \ldots, m, \tag{4}$$

where $d_k \in \mathbb{R}$. According to [2] this mechanical system forms a Hessenberg DAE of index-3. In other words, since this ODE system describes Newton's second law of motion relating generalized accelerations $\ddot{\mathbf{s}}$ (i.e., second derivatives of generalized positions) to forces, two differentiations of constraints (4) with respect to time t leads to saddle point system (3).

Further, let us denote $\mathbf{M} = (m_{ij}) \in \mathbb{R}^{n\times n}$ as a symmetric positive definite generalized mass matrix and matrix $\mathbf{G} = (g_{kj}) \in \{-1, 0, 1\}^{m\times n}$ with entries of value -1, 0 or 1, assumed to have full row rank, i.e., the constraints (4) are linearly independent. Then the system (3) for unknown differential variables $\mathbf{s}$ and unknown algebraic variables $\boldsymbol{\lambda} = (-\lambda_1, \ldots, -\lambda_m)^{\mathrm{T}}$ can be written in a more transparent way as (see, e.g., [18])

$$\underbrace{\begin{bmatrix} \mathbf{M} & \mathbf{G}^{\mathrm{T}} \\ \mathbf{G} & \mathbf{O}_m \end{bmatrix}}_{:=\mathbf{A}} \begin{bmatrix} \ddot{\mathbf{s}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{o}_m \end{bmatrix}, \tag{5}$$

where $\mathbf{O}_m \in \mathbb{R}^{m\times m}$ is a zero matrix, $\mathbf{q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)^{\mathrm{T}}$ a vector of general forces and $\mathbf{o}_m \in \mathbb{R}^m$ a zero vector, respectively.

Note. In order to make further analysis clearer, at this point we emphasize again to the readers that the positive integer m denotes the number of constraints, while the functions m_{ij} represent the elements of the generalized mass matrix $\mathbf{M}$. The relevant notations are therefore not related to each other.

Following classical results on the saddle point systems [22], we are able to eliminate algebraic variables $\boldsymbol{\lambda}$ and reformulate DAE (5) as an ODE system of size $2n$ in closed form. Specifically, according to algebraic properties of the saddle point matrices [5], one can explicitly express the inverse of $\mathbf{A}$ from (5) as

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{G}\mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{G}\mathbf{M}^{-1} & \mathbf{S}^{-1} \end{bmatrix}, \tag{6}$$

where $\mathbf{S} = -\mathbf{G}\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ is the Schur complement of $\mathbf{M}$ in $\mathbf{A}$. Recall that $\mathbf{A}$ is non-singular if and only if $\mathbf{S}$ is non-singular, which is guaranteed by the

positive definiteness of $\mathbf{M}$ and the full row rank property of $\mathbf{G}$, see, e.g., [16]. Next, left-multiplication of (5) by $\mathbf{A}^{-1}$ leads to

$$\begin{bmatrix} \ddot{\mathbf{s}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{q} + \mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{G}\mathbf{M}^{-1}\mathbf{q} \\ -\mathbf{S}^{-1}\mathbf{G}\mathbf{M}^{-1}\mathbf{q} \end{bmatrix}, \tag{7}$$

and elimination of algebraic variables $\boldsymbol{\lambda}$ results into the following second order ODE system

$$\ddot{\mathbf{s}} = \mathbf{B}\mathbf{q}, \qquad \text{with} \quad \mathbf{B} = (\mathbf{I} - \boldsymbol{\Pi})\mathbf{M}^{-1}, \tag{8}$$

where $\mathbf{I} \in \mathbb{R}^{n\times n}$ is a unit matrix and $\boldsymbol{\Pi} = -\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{G}$ is a projection matrix, i.e., it satisfies an idempotent property

$$\boldsymbol{\Pi}^2 = -\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{G}(-\mathbf{M}^{-1})\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{G} = -\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{S}\mathbf{S}^{-1}\mathbf{G} = \boldsymbol{\Pi}. \tag{9}$$

In other words, the differential component of (7) can be written as $\ddot{\mathbf{s}} = (\mathbf{I} - \boldsymbol{\Pi})\ddot{\mathbf{u}}$, where $\ddot{\mathbf{u}}$ denotes the generalized accelerations of $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$, representing the solution (i.e., generalized position coordinates) of the unconstrained problem $\mathbf{M}\ddot{\mathbf{u}} = \mathbf{q}$.

Subsequently, the standard substitution

$$z_1 = s_1, \ \ldots, \ z_n = s_n, \qquad z_{n+1} = \dot{s}_1, \ \ldots, \ z_{2n} = \dot{s}_n$$

transforms (8) to the first order ODE system of size $2n$. In general case, if $\mathbf{B} = \mathbf{B}(t, \mathbf{s}, \dot{\mathbf{s}})$ and $\mathbf{q} = \mathbf{q}(t, \mathbf{s}, \dot{\mathbf{s}})$ we obtain

$$\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z}), \tag{10}$$

where $\mathbf{z} = (z_1, \ldots, z_{2n})^{\mathrm{T}}$ are unknown generalized body positions and their velocities with the corresponding mapping $\mathbf{f} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$, defined with the aid of $\mathbf{B}$ and $\mathbf{q}$. However, when $\mathbf{M}$ depends only on time t; and applied forces $\mathbf{q}$ are linear with respect to $\mathbf{s}$ and $\dot{\mathbf{s}}$, the system (10) becomes linear. More precisely, let

$$\mathbf{q} = \mathbf{V}\mathbf{s} + \mathbf{D}\dot{\mathbf{s}}, \quad \text{for} \quad \mathbf{V} = \mathbf{V}(t), \ \mathbf{D} = \mathbf{D}(t) \in \mathbb{R}^{n\times n}, \tag{11}$$

then $\mathbf{f}(t, \mathbf{z}) = \mathbf{H}\mathbf{z}$, where $\mathbf{H} = \mathbf{H}(t) \in \mathbb{R}^{2n\times 2n}$ is defined as

$$\mathbf{H} = \begin{bmatrix} \mathbf{O}_n & \mathbf{I} \\ \mathbf{B}\mathbf{V} & \mathbf{B}\mathbf{D} \end{bmatrix}. \tag{12}$$

Furthermore, the time independent case leads to the first order ODE system with constant coefficients, i.e.,

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}, \tag{13}$$

where the constant matrix $\mathbf{C} \in \mathbb{R}^{2n\times 2n}$ is defined as (12) and its spectral properties are described below, cf. [4].

Theorem 1. *Let* $\mathbf{M}$, $\mathbf{V}$, $\mathbf{D} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{G} \in \mathbb{R}^{m\times n}$ *be constant matrices. Moreover, let* $\mathbf{M}$, $\mathbf{V}$ *be non-singular and* $\operatorname{rank}(\mathbf{G}) = m$, $0 < m < n$. *Then the matrix* $\mathbf{C}$ *from (13), defined as (12), is non-diagonalizable. More precisely, zero is eigenvalue of* $\mathbf{C}$ *that occurs with algebraic multiplicity* $2m$, *while its geometric multiplicity, i.e., the dimension of the nullspace of* $\mathbf{C}$, *is* m.

Proof. At first we show that $\operatorname{rank}(\mathbf{B}) = n - m$, where $\mathbf{B}$ is defined in (8). From $\operatorname{rank}(\mathbf{G}) = m$ we get that Schur complement $\mathbf{S}$ from (6) is non-singular. Then, the full-rank factorization property implies that $\operatorname{rank}(\mathbf{\Pi}) = m$ in (8). Since $\mathbf{\Pi}$ is a projection matrix, $(\mathbf{I} - \mathbf{\Pi})$ is also a projection matrix and $\operatorname{rank}(\mathbf{I} - \mathbf{\Pi}) = n - m$. The desired property follows from

$$\operatorname{rank}(\mathbf{B}) = \operatorname{rank}((\mathbf{I} - \mathbf{\Pi})\mathbf{M}^{-1}) = \operatorname{rank}(\mathbf{I} - \mathbf{\Pi}).$$

Since $\mathbf{V}$ is non-singular, $\operatorname{rank}(\mathbf{BV}) = \operatorname{rank}(\mathbf{B})$, then a simple rearrangement of rows in $\mathbf{C}$ (see below)

$$\mathbf{C} = \begin{bmatrix} \mathbf{O}_n & \mathbf{I} \\ \mathbf{BV} & \mathbf{BD} \end{bmatrix} \sim \begin{bmatrix} \mathbf{BV} & \mathbf{BD} \\ \mathbf{O}_n & \mathbf{I} \end{bmatrix}$$

implies that $\operatorname{rank}(\mathbf{C}) = n - m + n = 2n - m$, i.e., zero is eigenvalue of $\mathbf{C}$ having geometric multiplicity $2n - \operatorname{rank}(\mathbf{C}) = m$.

Further, in order to prove that the zero eigenvalue has algebraic multiplicity $2m$, it is sufficient to show that there exist m generalized eigenvectors corresponding to this eigenvalue. In other words, we need to prove that $\operatorname{rank}(\mathbf{C}^2) = \operatorname{rank}(\mathbf{C}^3) = 2n - 2m$. Indeed, we have

$$\mathbf{C}^2 = \begin{bmatrix} \mathbf{O}_n & \mathbf{I} \\ \mathbf{BV} & \mathbf{BD} \end{bmatrix} \begin{bmatrix} \mathbf{O}_n & \mathbf{I} \\ \mathbf{BV} & \mathbf{BD} \end{bmatrix} = \begin{bmatrix} \mathbf{BV} & \mathbf{BD} \\ \mathbf{BDBV} & \mathbf{BV} + \mathbf{BDBD} \end{bmatrix},$$

and left-multiplication by a lower triangular matrix with ones on the diagonal preserves the nullspace of $\mathbf{C}^2$, i.e.,

$$\begin{bmatrix} \mathbf{I} & \mathbf{O}_n \\ -\mathbf{BD} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{BV} & \mathbf{BD} \\ \mathbf{BDBV} & \mathbf{BV} + \mathbf{BDBD} \end{bmatrix} = \begin{bmatrix} \mathbf{BV} & \mathbf{BD} \\ \mathbf{O}_n & \mathbf{BV} \end{bmatrix}.$$

The last matrix in the equation above can be decomposed into a following product

$$\begin{bmatrix} \mathbf{BV} & \mathbf{BD} \\ \mathbf{O}_n & \mathbf{BV} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{O}_n \\ \mathbf{O}_n & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \mathbf{D} \\ \mathbf{O}_n & \mathbf{V} \end{bmatrix},$$

where the first factor is a block-diagonal matrix with blocks having rank of $n - m$ and the second factor is a non-singular matrix under $\operatorname{rank}(\mathbf{V}) = n$. Hence, one concludes that $\operatorname{rank}(\mathbf{C}^2) = 2(n - m) = 2n - 2m$.

Using the similar steps for $\mathbf{C}^3$, we find that

$$\begin{bmatrix} \mathbf{I} & \mathbf{O}_n \\ -\mathbf{BD} & \mathbf{I} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{O}_n & \mathbf{I} \\ \mathbf{BV} & \mathbf{BD} \end{bmatrix}^3}_{\mathbf{C}^3} = \begin{bmatrix} \mathbf{BDBV} & \mathbf{BV} + \mathbf{BDBD} \\ \mathbf{BVBV} & \mathbf{BVBD} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{BV} & \mathbf{BDB} \\ \mathbf{O}_n & \mathbf{BVB} \end{bmatrix}}_{\mathbf{W}} \begin{bmatrix} \mathbf{O}_n & \mathbf{I} \\ \mathbf{V} & \mathbf{D} \end{bmatrix},$$

and since the last matrix in the equation above is non-singular, it remains to prove that $\operatorname{rank}(\mathbf{W}) = 2n - 2m$. To cope with this, we proceed as follows.

Let $\mathbf{Q} \in \mathbb{R}^{n \times n}$ be a transformation matrix that performs elementary steps of the Gaussian elimination procedure to obtain a row echelon form of matrix $\mathbf{B}$, i.e.,

$$\mathbf{QB} = \begin{bmatrix} \widetilde{\mathbf{B}} \\ \mathbf{O}_{\mathbf{B}} \end{bmatrix},$$

where $\widetilde{\mathbf{B}} \in \mathbb{R}^{(n-m)\times n}$ is a reduced row echelon variant and $\mathbf{O_B} \in \mathbb{R}^{m\times n}$ is a zero matrix. By the definition, $\mathbf{Q}$ is non-singular and its multiplying by blocks $\mathbf{BV}$ and $\mathbf{BDB}$ leads to

$$\mathbf{QBV} = \begin{bmatrix} \widetilde{\mathbf{B}} \\ \mathbf{O_B} \end{bmatrix} \mathbf{V} = \begin{bmatrix} \widetilde{\mathbf{B}}\mathbf{V} \\ \mathbf{O_B} \end{bmatrix}, \qquad \mathbf{QBDB} = \begin{bmatrix} \widetilde{\mathbf{B}} \\ \mathbf{O_B} \end{bmatrix} \mathbf{DB} = \begin{bmatrix} \widetilde{\mathbf{B}}\mathbf{DB} \\ \mathbf{O_B} \end{bmatrix},$$

and

$$\mathbf{W} \sim \begin{bmatrix} \widetilde{\mathbf{B}}\mathbf{V} & \widetilde{\mathbf{B}}\mathbf{DB} \\ \mathbf{O_B} & \mathbf{O_B} \\ \mathbf{O}_n & \mathbf{BVB} \end{bmatrix}.$$

Since $\mathbf{V}$ is non-singular, $\widetilde{\mathbf{B}}\mathbf{V}$ has full row rank and it remains to show that $\operatorname{rank}(\mathbf{BVB}) = n - m$.

Let $\mathbf{R} \in \mathbb{R}^{n\times n}$ be a transformation matrix that performs elementary steps of the Gaussian elimination procedure to obtain a row echelon form of matrix $\mathbf{B}^{\mathrm{T}}$, i.e.,

$$(\mathbf{RB}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{BR}^{\mathrm{T}} = \begin{bmatrix} \widehat{\mathbf{B}} & \mathbf{O}_{\mathbf{B}}^{\mathrm{T}} \end{bmatrix},$$

where $\widehat{\mathbf{B}} \in \mathbb{R}^{n\times(n-m)}$ is a reduced column echelon variant. By the definition, $\mathbf{R}^{\mathrm{T}}$ is non-singular and it holds that $\operatorname{rank}(\mathbf{BVB}) = \operatorname{rank}(\mathbf{QBVBR}^{\mathrm{T}})$. Next, a simple calculation leads to

$$\mathbf{QBVBR}^{\mathrm{T}} = \begin{bmatrix} \widetilde{\mathbf{B}} \\ \mathbf{O_B} \end{bmatrix} \mathbf{V} \begin{bmatrix} \widehat{\mathbf{B}} & \mathbf{O}_{\mathbf{B}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{B}}\mathbf{V}\widehat{\mathbf{B}} & \mathbf{O}_{(n-m)\times m} \\ \mathbf{O}_{m\times(n-m)} & \mathbf{O}_m \end{bmatrix},$$

where $\mathbf{O}_{(n-m)\times m}$, $\mathbf{O}_{m\times(n-m)}$ are zero matrices of given sizes. Since $\mathbf{V}$ is non-singular, $\widetilde{\mathbf{B}}\mathbf{V}\widehat{\mathbf{B}} \in \mathbb{R}^{(n-m)\times(n-m)}$ is non-singular and one can easily find that $\operatorname{rank}(\mathbf{QBVBR}^{\mathrm{T}}) = \operatorname{rank}(\widetilde{\mathbf{B}}\mathbf{V}\widehat{\mathbf{B}}) = n - m$. This completes that $\operatorname{rank}(\mathbf{C}^3) = 2n - 2m$.

Finally, taking into account that $\mathbf{C}$ has zero eigenvalue, whose algebraic and geometric multiplicities are different, the matrix $\mathbf{C}$ is non-diagonalizable. □

In line with the theory of linear autonomous ODE systems, the (exact) general solution of (13) is given by a matrix exponential as

$$\mathbf{z}(t) = \mathbf{c} \cdot \exp(\mathbf{C}t), \quad t > 0, \tag{14}$$

where $\mathbf{c} \in \mathbb{R}^n$ is an arbitrary vector. The non-diagonalizable property of $\mathbf{C}$ makes the calculation of $\exp(\mathbf{C}t)$ more challenging. In this case, we decompose the matrix $\mathbf{C}$ as a sum $\mathbf{J} + \mathbf{N}$, where $\mathbf{J}$ is diagonalizable, $\mathbf{N}$ is nilpotent, and $\mathbf{JN} = \mathbf{NJ}$, see Jordan–Chevalley decomposition in [10]. Then, one can write

$$\exp(\mathbf{C}t) = \exp((\mathbf{J} + \mathbf{N})t) = \exp(\mathbf{J}t) \cdot \exp(\mathbf{N}t), \quad t > 0. \tag{15}$$

Since $\mathbf{N}$ is nilpotent, the series for $\exp(\mathbf{N}t)$ is finite. Nevertheless, the construction of this decomposition still remains the challenging for systems of large size. Therefore, a numerical approach for solving of such ODE problems is common, see Subsection 3.2 for the particular case.

2.2 Switchable Constraints

In the second scenario (with predefined settings), switchable constraints are included. More precisely, we assume that l constraints are switched off, $1 \leq l \leq m$, it means that matrix $\mathbf{G}$ has l zero rows on appropriate places corresponding to the order of the disabled constraints. Since $\mathrm{rank}(\mathbf{G}) = m - l$, the Schur complement $\mathbf{S}$ of $\mathbf{M}$ in $\mathbf{A}$ from (5) is not invertible. However, we can follow the same algorithm using Moore-Penrose inverse (i.e., a pseudoinverse).

Definition 1 (see [11])**.** Let $\mathbf{X} \in \mathbb{R}^{r\times s}$, a pseudoinverse of $\mathbf{X}$ is defined as a matrix $\mathbf{X}^+ \in \mathbb{R}^{s\times r}$ satisfying all of the following four criteria, known as the Moore–Penrose conditions:

(i) $\mathbf{X}\mathbf{X}^+\mathbf{X} = \mathbf{X}$,

(ii) $\mathbf{X}^+\mathbf{X}\mathbf{X}^+ = \mathbf{X}^+$,

(iii) $(\mathbf{X}\mathbf{X}^+)^{\mathrm{T}} = \mathbf{X}\mathbf{X}^+$,

(iv) $(\mathbf{X}^+\mathbf{X})^{\mathrm{T}} = \mathbf{X}^+\mathbf{X}$.

This generalization of the inverse matrix produces a matrix that is close to the inverse matrix in some sense, but is computed for matrices that are not invertible. Moreover, for any matrix $\mathbf{X}$ there is one and only one pseudoinverse $\mathbf{X}^+$, see [11].

Accordingly, such a pseudoinverse of $\mathbf{S}$ is defined as a matrix $\mathbf{S}^+$ satisfying the above Moore–Penrose conditions. As a result of that, this approach preserves the matrix block-wise calculations as in the non-singular case of the Schur complement $\mathbf{S}$ and allows us to proceed through the same steps (6)–(9) with matrix $\mathbf{S}^+$ instead of $\mathbf{S}^{-1}$. In accordance with the above, similarly to Theorem 1, we are able to formulate its extension to the case of l (selected) disabled constraints.

Corollary. *If* $\mathrm{rank}(\mathbf{G}) = m - l$ *and* $l < m$*, then the statements of Theorem 1 remain valid with the difference that the zero eigenvalue of* $\mathbf{C}$ *has the algebraic multiplicity* $2(m - l)$ *and the geometric multiplicity* $m - l$*.*

Remark. For unconstrained problem (i.e., $l = m$) zero is not eigenvalue of $\mathbf{C}$ and the (eventual) non-diagonalizable property is driven by spectral properties of matrices $\mathbf{V}$ and $\mathbf{D}$. However, it goes beyond the scope of the related issue.

The methodology presented above will be illustrated on the idealized two-mass oscillator with a switchable constraint (between two masses) that represents the simplest, deeply known, scenario to highlight the functionality of this methodology. With respect to the practical point of view, this problem reminds the quarter car model with friction or the lock spring-damper element.

3. Two-mass Oscillator with Switchable Constraint

Accordingly the notation introduced, we set $n = 2$, $m = 1$ and consider that the mechanical system is described by its kinetic and potential energy as follows

$$\mathcal{K}(\dot{\mathbf{s}}) = \frac{1}{2}m_1\dot{s}_1^2 + \frac{1}{2}m_2\dot{s}_2^2, \tag{16}$$

$$\mathcal{V}(\mathbf{s}) = \frac{1}{2}k_1 s_1^2 + \frac{1}{2}k_2 s_2^2, \tag{17}$$

where positive constants m_1, m_2 are masses of the rigid bodies and positive constants k_1, k_2 represent corresponding stiffness. Moreover, to be as clear as possible, we omit external forces and dissipative components (e.g., dumping), i.e., $\overline{\mathcal{Q}} = 0$ and $\mathcal{D} = 0$, respectively. Furthermore, we consider a simple condition that both rigid bodies of the mechanical system are distant (from each other) by a fixed length for the entire time that the constraint is active.

Taking these assumptions into account, the motion of such mechanical system is described by the following ODE system

$$m_1 \ddot{s}_1(t) = -k_1 s_1(t), \tag{18}$$
$$m_2 \ddot{s}_2(t) = -k_2 s_2(t), \tag{19}$$

subjected to the linear constraint

$$s_1(t) - s_2(t) = d_1, \qquad d_1 \in \mathbb{R}. \tag{20}$$

Double differentiation of the constraint (20) yields a constraint on the acceleration level

$$\ddot{s}_1(t) - \ddot{s}_2(t) = 0, \tag{21}$$

and using the approach of Lagrange multipliers we rewrite (18)–(20) as the well-known saddle point system

$$\begin{bmatrix} \mathbf{M} & \mathbf{G}^{\mathrm{T}} \\ \mathbf{G} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{s}} \\ -\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{V}\mathbf{s} \\ 0 \end{bmatrix}, \tag{22}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{23}$$

and $\lambda \in \mathbb{R}$ is the Lagrange multiplier.

To uniquely solve the problem (22)–(23), it is necessary to add a couple of initial conditions. The setting of initial conditions to determine the particular solution of DAEs is a delicate issue. By default, we need to prescribe $2n$ independent initial conditions for the second ODE system (18)–(19) in the form

$$s_1(0) = \alpha, \quad s_2(0) = \beta, \quad \dot{s}_1(0) = \gamma, \quad \dot{s}_2(0) = \delta, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \tag{24}$$

On the other hand, from constraint (20), one can deduce that conditions (24) have to be consistent. Therefore, it is necessary to prescribe only n conditions

$$s_1(0) = \alpha, \qquad \dot{s}_1(0) = \gamma, \tag{25}$$

and the constraint (20) and its differentiation lead to

$$s_2(0) = s_1(0) - d_1, \qquad \dot{s}_2(0) = \dot{s}_1(0). \tag{26}$$

Within the forthcoming Subsections 3.1–3.2 we present analytical and numerical approach to solve the unconstrained and constrained ODE system (18)–(19).

3.1 Quantitative Properties and ODE Analysis

At first, we consider the unconstrained system, represented by separated ODEs (18) and (19), rewritten as

$$\ddot{s}_j(t) + \frac{k_j}{m_j} s_j(t) = 0, \qquad j = 1, 2. \tag{27}$$

Following the theory of the second order ODEs with constant coefficients [13], one can express the exact solutions of (27) in the following forms

$$s_1(t) = c_1 \cos\left(\sqrt{\frac{k_1}{m_1}} \cdot t\right) + c_3 \sin\left(\sqrt{\frac{k_1}{m_1}} \cdot t\right), \tag{28}$$

$$s_2(t) = c_2 \cos\left(\sqrt{\frac{k_2}{m_2}} \cdot t\right) + c_4 \sin\left(\sqrt{\frac{k_2}{m_2}} \cdot t\right), \tag{29}$$

where constants $c_1, \ldots, c_4$ depend on initial states (24). Specifically,

$$c_1 = \alpha, \qquad c_2 = \beta, \qquad c_3 = \gamma\sqrt{\frac{m_1}{k_1}}, \qquad c_4 = \delta\sqrt{\frac{m_2}{k_2}}. \tag{30}$$

Note, while the initial states affect the magnitude of an amplitude of (28) and (29), the ratio $k_j : m_j$ controls the corresponding frequency $f_j = \frac{1}{2\pi}\sqrt{\frac{k_j}{m_j}}$, $j = 1, 2$, see Subsection 4.1 for the particular case.

Secondly, we consider the constrained system (27) with (20), represented in line with the presented methodology by the first order system (13), rewritten component-wise as follows

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \\ \dot{z}_4(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b_{11}k_1 & -b_{12}k_2 & 0 & 0 \\ -b_{21}k_1 & -b_{22}k_2 & 0 & 0 \end{bmatrix}}_{:=\mathbf{C}} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix}, \tag{31}$$

where $\mathbf{B} = (b_{ij}) \in \mathbb{R}^{2\times 2}$ is given by (8) and $\operatorname{rank}(\mathbf{B}) = 1$.

Following the theory of the systems of first order ODEs with constant coefficients [13], the general solution of (31) can be expressed as a linear combination of fundamental solutions, i.e.,

$$\mathbf{z}(t) = c_1\boldsymbol{\psi}_1(t) + c_2\boldsymbol{\psi}_2(t) + c_3\boldsymbol{\psi}_3(t) + c_4\boldsymbol{\psi}_4(t), \quad c_1, \ldots, c_4 \in \mathbb{R}, \tag{32}$$

where vector-valued functions $\boldsymbol{\psi}_j(t)$, $j = 1, \ldots, 4$ are linearly independent. In order to construct these fundamental solutions, we compute the eigenvalues and the corresponding eigenvectors of the system matrix $\mathbf{C}$ from (31), i.e.,

$$\ell_{1,2} = \pm i\sqrt{\frac{k_1 + k_2}{m_1 + m_2}}, \quad \ell_{3,4} = 0, \quad \mathbf{w}_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm i\sqrt{\frac{m_1+m_2}{k_1+k_2}} \\ \pm i\sqrt{\frac{m_1+m_2}{k_1+k_2}} \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} k_2 \\ -k_1 \\ 0 \\ 0 \end{bmatrix}. \tag{33}$$

According to Theorem 1, the system matrix $\mathbf{C}$ is non-diagonalizable and the eigenvalue $\ell_{3,4} = 0$ occurs with algebraic multiplicity 2, but for it there are not 2 linearly independent eigenvectors. Therefore, we append a generalized eigenvector $\mathbf{w}_4$ as a solution of $\mathbf{C}\mathbf{w}_4 = \mathbf{w}_3$ to ensure the validity of the terms $\mathbf{C}^2\mathbf{w}_4 = (0,0,0,0)^{\mathrm{T}}$ and $\mathbf{C}\mathbf{w}_4 \neq (0,0,0,0)^{\mathrm{T}}$. Simple calculations lead to $\mathbf{w}_4 = (0,0,k_2,-k_1)^{\mathrm{T}}$. Then taking into account a complex conjugate property (i.e., $\ell_2 = \overline{\ell_1}$ and $\mathbf{w}_2 = \overline{\mathbf{w}_1}$) and multiple eigenvalue ($\ell_{3,4} = 0$), we obtain

$$\begin{aligned} \boldsymbol{\psi}_1(t) &= \Re\left(\exp(\ell_1 t)\mathbf{w}_1\right), \qquad \boldsymbol{\psi}_2(t) = \Im\left(\exp(\ell_1 t)\mathbf{w}_1\right), \\ \boldsymbol{\psi}_3(t) &= \mathbf{w}_3, \qquad \boldsymbol{\psi}_4(t) = t \cdot \mathbf{w}_3 + \mathbf{w}_4. \end{aligned} \tag{34}$$

Next, we insert (34) into (32), then the constrained solutions $s_1(t) = z_1(t)$ and $s_2(t) = z_2(t)$ have the following form

$$s_1(t) = c_1 \cos\left(\sqrt{\frac{k_1+k_2}{m_1+m_2}} \cdot t\right) + c_2 \sin\left(\sqrt{\frac{k_1+k_2}{m_1+m_2}} \cdot t\right) + (c_3 + c_4 t)k_2, \tag{35}$$

$$s_2(t) = c_1 \cos\left(\sqrt{\frac{k_1+k_2}{m_1+m_2}} \cdot t\right) + c_2 \sin\left(\sqrt{\frac{k_1+k_2}{m_1+m_2}} \cdot t\right) - (c_3 + c_4 t)k_1. \tag{36}$$

One can easily resolve that both solutions (35)–(36) have the same frequency $f_1 = f_2 = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m_1+m_2}}$. Concerning the constants $c_1, \ldots, c_4$ in (35)–(36), from (20) and its differentiation (i.e., $\dot{s}_1(t) = \dot{s}_2(t)$) we get

$$c_3 = \frac{d_1}{k_1+k_2}, \qquad c_4 = 0. \tag{37}$$

Finally, the condition (25) results to

$$c_1 = \alpha - c_3 k_2 = \alpha - d_1 + c_3 k_1, \qquad c_2 = \gamma\sqrt{\frac{m_1+m_2}{k_1+k_2}}. \tag{38}$$

Remark. The result (35)–(36) can be alternatively obtained by summing the equations (27) and using (20) to obtain the one ODE of the second order with the non-homogeneous right-hand side. It is left to the interested reader to find a solution this way.

3.2 Numerical Approach

The simplest numerical methods for solving DAEs are variants of well-known time integration schemes for ODEs that are more or less directly applied. However, this approach are not successful in general, see [12].

Regarding the illustrative experiment it is possible to use a direct time integration, for which responses of displacements and velocities are calculated using step-by-step integration in time without modifications to the equations of motion as long as switchable constraints remain unchanged. In particular, explicit schemes have become one of the most widely used classes of direct methods, especially for the analysis of transient phenomena such as switching constraints. Nevertheless,

the disadvantage of the explicit treatment is that the resulting schemes are conditionally stable, i.e., the time step size has to be below a critical value, see [2].

Among the explicit schemes, Runge-Kutta methods are very popular. These single-step methods use a series of iterative calculations to approximate the solution at each time level within a prespecified time interval $[0, T]$, with higher-order techniques to provide more accurate results while cost the per time step is small.

In what follows, we present the solution to (31) with the given initial state (25)–(26) using a particular Runge-Kutta scheme. Consider the following time partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{r-1} < t_r = T \tag{39}$$

with fixed time step $\tau = t_{i+1} - t_i$ (for simplicity). However, it is also possible to use an adaptive time step, and a simple generalization of the procedure below is left to the interested reader. Then the single-step procedure of problem (31) reads: Find $\mathbf{z}^{(i+1)} \approx \mathbf{z}(t_{i+1})$ such as

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}, \qquad \text{with} \quad \mathbf{z}(t_i) = \mathbf{z}^{(i)}. \tag{40}$$

Next, we recall the classic Runge-Kutta (fourth-order) method [12], defined by recurrent formulae related to (40), i.e.,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{C}\mathbf{z}^{(i)}, \qquad \mathbf{r}_2 = \mathbf{C}\left(\mathbf{z}^{(i)} + \frac{\tau}{2}\mathbf{r}_1\right), \\ \mathbf{r}_3 &= \mathbf{C}\left(\mathbf{z}^{(i)} + \frac{\tau}{2}\mathbf{r}_2\right), \qquad \mathbf{r}_4 = \mathbf{C}\left(\mathbf{z}^{(i)} + \tau\mathbf{r}_3\right), \end{aligned} \tag{41}$$

and

$$\mathbf{z}^{(i+1)} = \frac{\tau}{6}\left(\mathbf{r}_1 + 2\mathbf{r}_2 + 2\mathbf{r}_3 + \mathbf{r}_4\right). \tag{42}$$

Putting (41) into (42), then after a simple rearrangement, we get

$$\mathbf{z}^{(i+1)} = \mathbf{z}^{(i)} + \tau\mathbf{C}\mathbf{z}^{(i)} + \frac{1}{2}\tau^2\mathbf{C}^2\mathbf{z}^{(i)} + \frac{1}{6}\tau^3\mathbf{C}^3\mathbf{z}^{(i)} + \frac{1}{24}\tau^4\mathbf{C}^4\mathbf{z}^{(i)}. \tag{43}$$

In other words, since the general solution is given by a matrix exponential (14), the exact solution to (40) takes the form $\mathbf{z}^{(i+1)} = \mathbf{z}^{(i)}\exp(\mathbf{C}\tau)$ and the first five terms of the corresponding power series give (43). Therefore, to improve the classic Runge-Kutta scheme, one can use the approximation of the pth order of the corresponding matrix exponential as

$$\mathbf{z}^{(i+1)} = \sum_{j=0}^{p} \frac{\tau^j}{j!}\mathbf{C}^j\mathbf{z}^{(i)}, \tag{44}$$

or directly compute the matrix exponential $\exp(\mathbf{C}\tau)$ as such itself.

Finally, note that for the time dependent case $\mathbf{C} = \mathbf{C}(t)$, one has to rely on numerical scheme, e.g., (42) where (41) take the following forms, i.e.,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{C}(t_i)\mathbf{z}^{(i)}, \qquad \mathbf{r}_2 = \mathbf{C}\left(t_i + \frac{\tau}{2}\right)\left(\mathbf{z}^{(i)} + \frac{\tau}{2}\mathbf{r}_1\right), \\ \mathbf{r}_3 &= \mathbf{C}\left(t_i + \frac{\tau}{2}\right)\left(\mathbf{z}^{(i)} + \frac{\tau}{2}\mathbf{r}_2\right), \qquad \mathbf{r}_4 = \mathbf{C}\left(t_i + \tau\right)\left(\mathbf{z}^{(i)} + \tau\mathbf{r}_3\right). \end{aligned} \tag{45}$$

Remark. We proceed through the same steps above also for the unconstrained case, where $\mathbf{B} = \mathbf{M}^{-1}$ in (31).

4. Numerical Simulations

In this section, we present numerical experiments on selected scenarios of two-mass oscillator with a switchable constraint (18)–(20) in order to verify the validity of the presented methodology and illustrate its usage. This two-mass model is used for the most straightforward simulation of a quarter car suspension system. In our case, the suspension element includes dry friction or active control capable of locking the suspension movement.

In the beginning, we deal with the single stage (i.e., the constrained or the unconstrained system), representing the initial setting. Secondly, to provide a brief insight to the transient behavior, we investigate constraint violation as well as its activation in the two-phase case. Furthermore, from the practical point of view, we perform an intentional control as a repeated (de-)activation of a switchable constraint. Finally, we extend simulations to the damped case of intentional control.

To be consistent in all cases, we consider fixed mass and stiffness data

$$m_1 = 10, \qquad m_2 = 1, \qquad k_1 = 10, \qquad k_2 = 5. \tag{46}$$

The used numerical scheme (44) is implemented through the MATLAB platform with $p = 4$ and $\tau = 0.01$. Note that these parameters seem to be appropriately chosen for the experimental study considered and taking a higher order p or a smaller time step τ will not improve the results significantly.

4.1 Initial Stage

We start with the unconstrained system (18)–(19) subjected to the following initial conditions

$$s_1(0) = 0.5, \quad s_2(0) = 0, \quad \dot{s}_1(0) = 0, \quad \dot{s}_2(0) = 1. \tag{47}$$

The (time) development of the corresponding components of positions, velocities and accelerations is depicted in Fig. 1. One can easily observe the different frequencies in graphs of position, velocity and acceleration components. More precisely, application of (28)–(30) leads to

$$s_1(t) = 0.5 \cdot \cos(t), \qquad s_2(t) = \frac{\sqrt{5}}{5} \cdot \sin\left(\sqrt{5} \cdot t\right), \qquad t > 0, \tag{48}$$

with the different frequencies $f_1 = \frac{1}{2\pi}$ and $f_2 = \frac{\sqrt{5}}{2\pi}$.

Secondly, we consider the constrained system (18)–(20) with the following data

$$s_1(0) = 0.5, \qquad \dot{s}_1(0) = 0, \qquad d_1 = 0.5. \tag{49}$$

From Fig. 2 one can recognize the identical graphs of velocity and acceleration components and shifted graphs (by d_1) of position components. These observations are in a good agreement with analytical solutions (35)–(38), i.e.,

$$s_1(t) = \frac{1}{3}\cos\left(\sqrt{\frac{15}{11}} \cdot t\right) + \frac{1}{6}, \qquad s_2(t) = \frac{1}{3}\cos\left(\sqrt{\frac{15}{11}} \cdot t\right) - \frac{1}{3}, \qquad t > 0, \tag{50}$$

with the same frequencies $f_1 = f_2 = \frac{1}{2\pi}\sqrt{\frac{15}{11}}$.

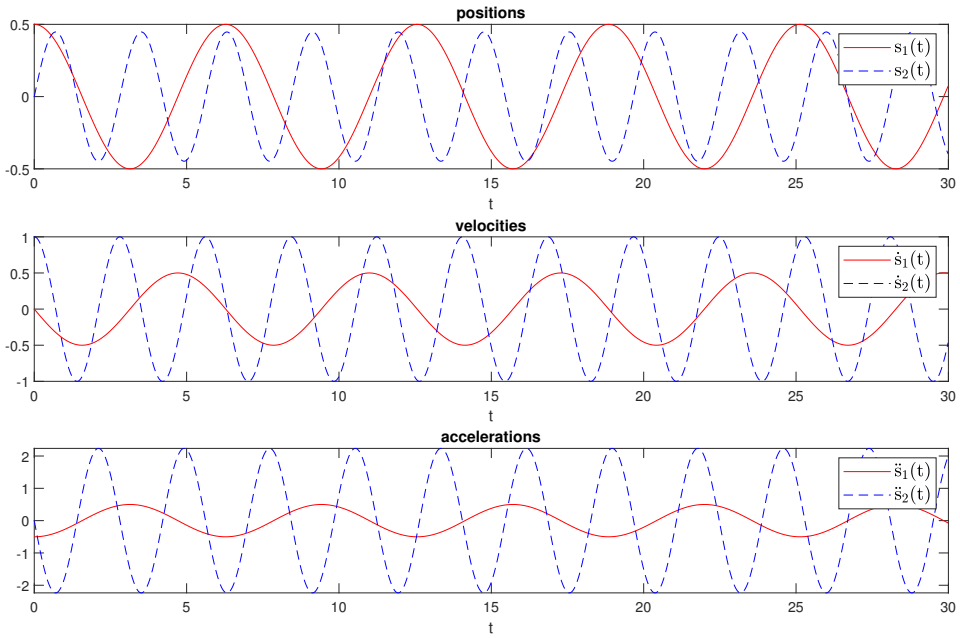

Fig. 1 *Initial stage: Position, velocity and acceleration components of the unconstrained two-mass oscillator.*

4.2 Two-phase Case

As the first scenario we take the constrained system (18)–(20) with data (49), where the constraint (20) is violated at fixed time instant $t = T_{\mathrm{D}} > 0$. This situation can be represented incorporating the piecewise constant matrix (in general)

$$\mathbf{G} = \mathbf{G}(t) = \begin{cases} \begin{bmatrix} 1 & -1 \end{bmatrix}, & \text{if} \quad 0 \le t \le T_{\mathrm{D}}, \\ \begin{bmatrix} 0 & 0 \end{bmatrix}, & \text{if} \quad t > T_{\mathrm{D}}. \end{cases} \tag{51}$$

Since (51) is not differentiable at $t = T_{\mathrm{D}}$, we divide the whole system in two subproblems linked via smoothing conditions

$$\lim_{t \to T_{\mathrm{D}}-} \mathbf{s}(t) = \mathbf{s}(T_{\mathrm{D}}) = \lim_{t \to T_{\mathrm{D}}+} \mathbf{s}(t), \qquad \lim_{t \to T_{\mathrm{D}}-} \dot{\mathbf{s}}(t) = \dot{\mathbf{s}}(T_{\mathrm{D}}) = \lim_{t \to T_{\mathrm{D}}+} \dot{\mathbf{s}}(t), \tag{52}$$

and treat both subproblems (separately) in sequential order. This approach leads to a discontinuity in $\ddot{\mathbf{s}}(t)$ at $t = T_{\mathrm{D}}$. More precisely, from (8) and (51) we get one-sided limits

$$\lim_{t \to T_{\mathrm{D}}-} \ddot{\mathbf{s}}(t) = (\mathbf{I} - \mathbf{\Pi})\mathbf{M}^{-1}\mathbf{V}\mathbf{s}(T_{\mathrm{D}}), \tag{53}$$

$$\lim_{t \to T_{\mathrm{D}}+} \ddot{\mathbf{s}}(t) = \mathbf{M}^{-1}\mathbf{V}\mathbf{s}(T_{\mathrm{D}}), \tag{54}$$

which are not equal in general.

The particular scenario for (49) and $T_{\mathrm{D}} = 12$ is shown in Fig. 3. The behavior during the first phase is identical to Fig. 2. However, at the decoupling time $t = T_{\mathrm{D}}$,

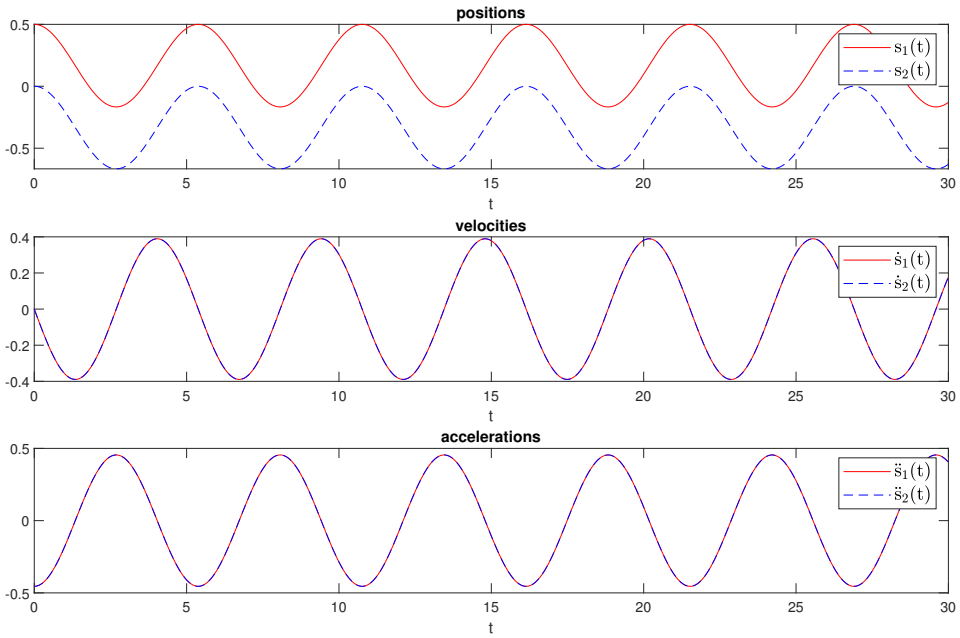

Fig. 2 *Initial stage: Position, velocity and acceleration components of the constrained two-mass oscillator.*

the discontinuity in both acceleration components is well resolved, while position and velocity components remain continuous throughout the simulation. Moreover, this disconnection brings a change in frequencies.

Compared to the decoupling scenario the issue of a constraint activation is more delicate. The reason is that the connection is possible only at certain time instants, when the velocity components of both rigid bodies are equal. To find the intersections of both velocity graphs, we define a relative velocity $\dot{s}_{\mathrm{rel}}(t) = \dot{s}_1(t) - \dot{s}_2(t)$ and localize its zero nodes, i.e.,

$$\dot{s}_{\mathrm{rel}}(t) = 0 \Longleftrightarrow \dot{s}_1(t) = \dot{s}_2(t). \tag{55}$$

Since $\dot{s}_{\mathrm{rel}}$ is evaluated in a discrete way during the simulation procedure, we simply identify the changing signs and use an t-intercept of the corresponding linear interpolant to precise the value of T_{C}, for which (55) holds.

In what follows, consider an unconstrained system (18)–(19), i.e., with $\mathbf{G} = [0 \;\; 0]$, and let $T_{\mathrm{D}}^{\mathrm{rel}}$ be a (minimal) user-defined relevant time of the first (disconnected) phase, then we find $T_{\mathrm{C}} > T_{\mathrm{D}}^{\mathrm{rel}}$ as small as possible to satisfy (55). Next, we redefine $\mathbf{G} = [1 \;\; -1]$ for all $t > T_{\mathrm{C}}$ and introduce the constrained system (18)–(20) with

$$d_1 = \lim_{t \to T_{\mathrm{C}}-} s_1(t) - s_2(t). \tag{56}$$

Similarly as in (52), we require the continuity of $\mathbf{s}(t)$ and $\dot{\mathbf{s}}(t)$ at $t = T_{\mathrm{C}}$. Again,

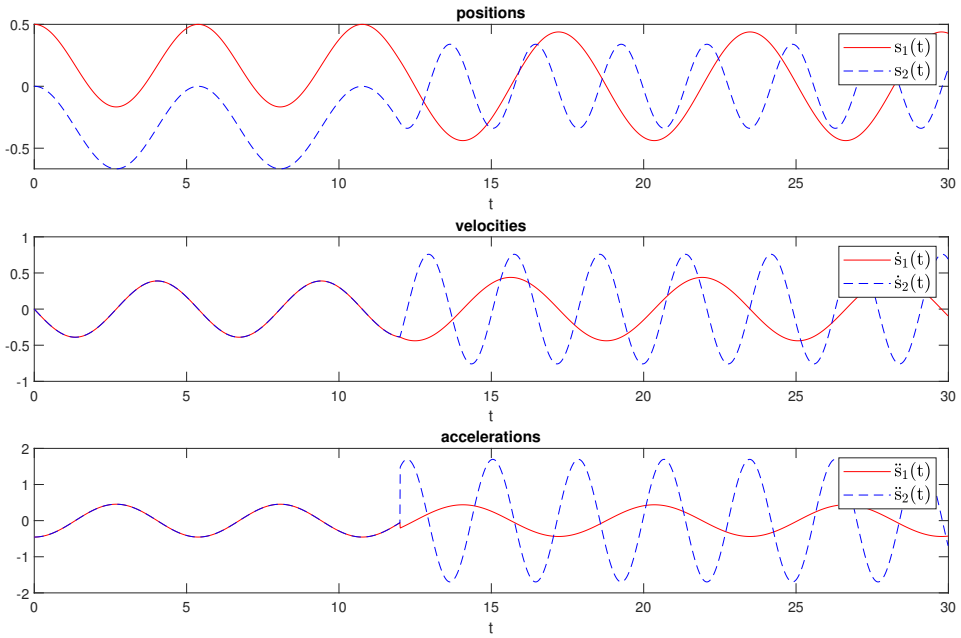

Fig. 3 *Decoupling: Position, velocity and acceleration components of the two-mass oscillator with a violated constraint.*

we obtain a discontinuity in $\ddot{\mathbf{s}}(t)$ at $t = T_{\mathrm{C}}$, because one-sided limits

$$\lim_{t \to T_{\mathrm{C}}-} \ddot{\mathbf{s}}(t) = \mathbf{M}^{-1}\mathbf{V}\mathbf{s}(T_{\mathrm{C}}), \tag{57}$$

$$\lim_{t \to T_{\mathrm{C}}+} \ddot{\mathbf{s}}(t) = (\mathbf{I} - \mathbf{\Pi})\mathbf{M}^{-1}\mathbf{V}\mathbf{s}(T_{\mathrm{C}}) \tag{58}$$

are not equal in general.

The particular scenario for (47) and $T_{\mathrm{D}}^{\mathrm{rel}} = 10$ is captured in Fig. 4. The value of T_{C} was set approximately as 10.76. As expected, the behavior is completely reversed as in Fig. 3. Specifically, the first phase is identical to Fig. 1 and, for $t > T_{\mathrm{C}}$, the frequencies of all components are the same and the difference of position components is constant (i.e., the relative velocity is zero).

4.3 Intentional Control

The following experiment extends the two-phase case in terms of artificial control. Consider the system (18)–(19) with repeated activation and violation of the constraint (20). Without loss of generality, we start from the unconstrained system. Relevant changes of the constraint (20) are driven by parameters $T_{\mathrm{D}}^{\mathrm{rel}}$ and $T_{\mathrm{C}}^{\mathrm{rel}}$, where the first one represents a (minimal) user-defined relevant period of the disconnected phase and the second one an (exact) user-defined relevant period of the connected phase. Further, following the approach from Subsection 4.2 with only

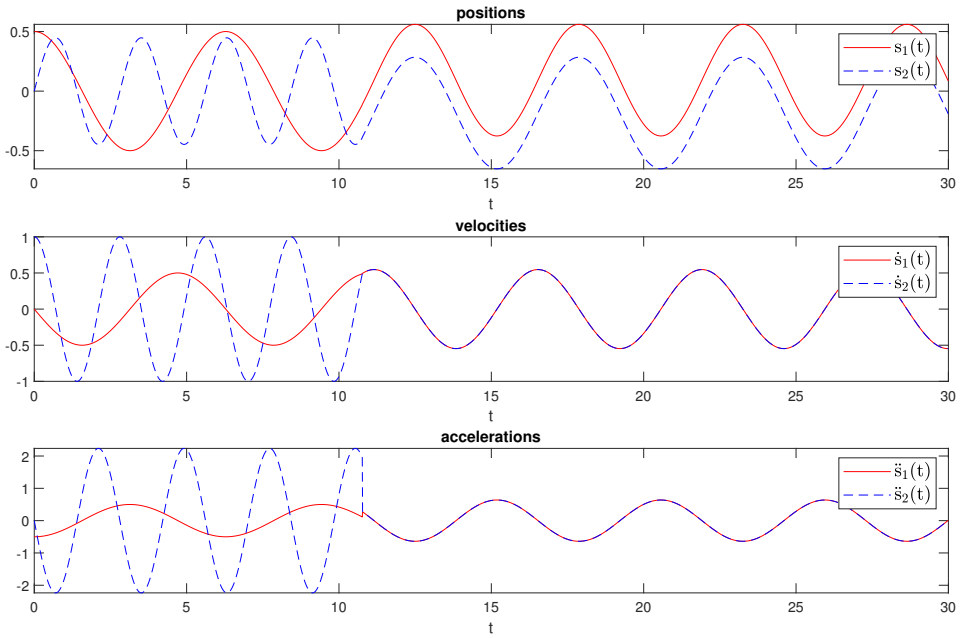

Fig. 4 *Coupling: Position, velocity and acceleration components of the two-mass oscillator with an activated constraint.*

one transient level, we obtain sets $\{T_{\mathrm{C}}^{(i)}\}$ and $\{T_{\mathrm{D}}^{(i)}\}$ satisfying

$$T_{\mathrm{C}}^{(i)} >= T_{\mathrm{D}}^{(i-1)} + T_{\mathrm{D}}^{\mathrm{rel}}, \qquad T_{\mathrm{D}}^{(i)} = T_{\mathrm{C}}^{(i)} + T_{\mathrm{C}}^{\mathrm{rel}}, \quad i = 1, 2, \ldots, \tag{59}$$

where $T_{\mathrm{D}}^{(0)} = 0$ and $T_{\mathrm{C}}^{(i)}$ is as small as possible to satisfy the relation above and (55). Basically, the time instants (59) determine the intentional control of the constraint (20) and as a result the entire oscillator. The behavior of such oscillator with a switchable constraint subjected to the intentional control with $T_{\mathrm{D}}^{\mathrm{rel}} = T_{\mathrm{C}}^{\mathrm{rel}} = 1$ and data (47) is depicted in Fig. 5. At first glance the time development of the position components may seem random, however, graphs of velocity and acceleration components clearly identify transient levels and different phases, see Fig. 5 (upper). This behavior is much more apparent from the relative values as constant and non-constant phases in Fig. 5 (bottom).

From a technical point of view, as a quarter car suspension system is concerned, the locking or unlocking can be done by the intervention of an active system, which is particularly useful in various practical situations where a vehicle has to respect specific requirements. The detection of the lock state can be performed by the accelerometers on the sprung/unsprung masses, and the derived vertical velocities of these components are identical. Conversely, the unlock state results in different vertical velocities. Potential applications can be found in commercial, experimental, heavy terrain, and military vehicles.

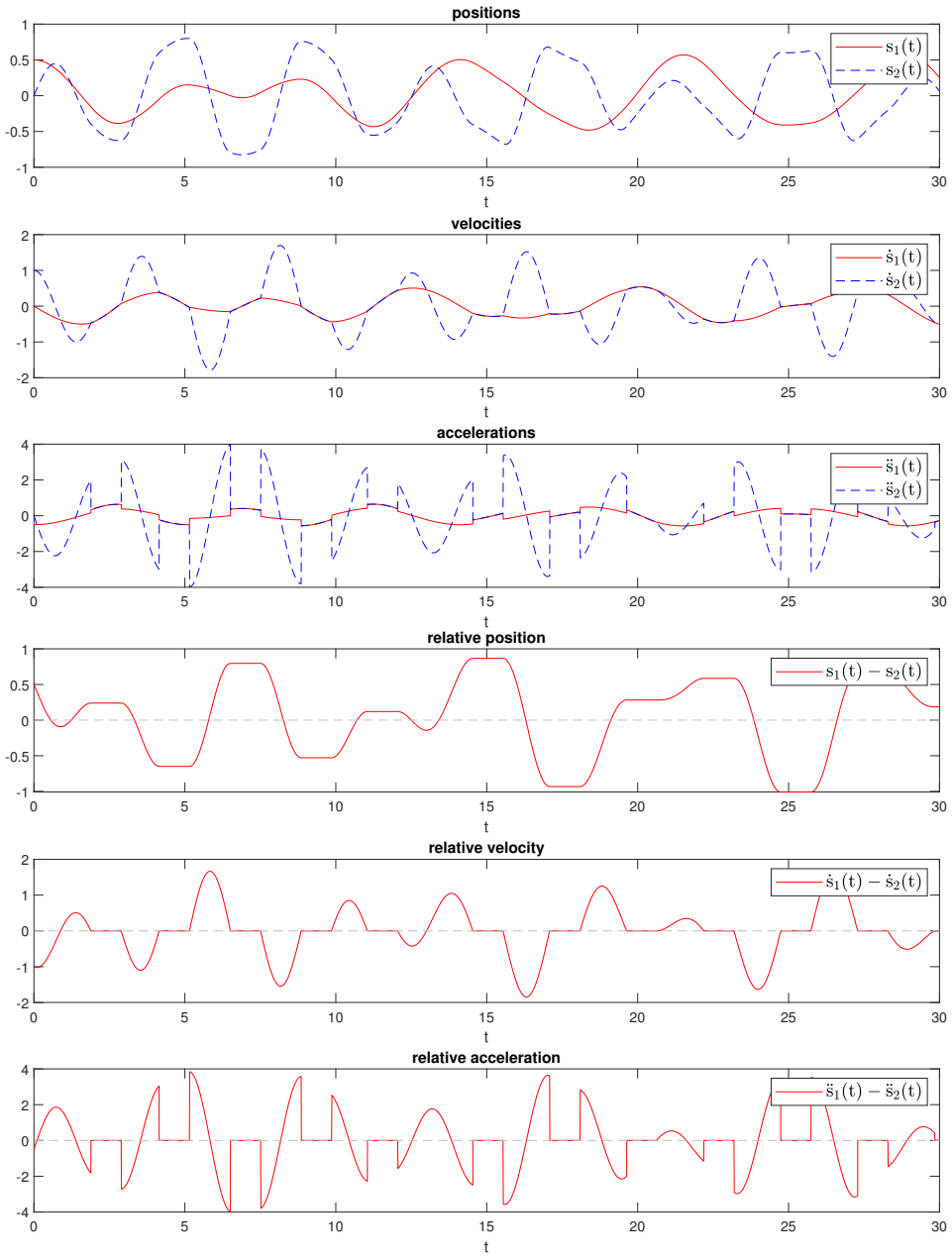

Fig. 5 *Intentional control: Position, velocity and acceleration components of the two-mass oscillator (upper) and the corresponding differences of these components (bottom).*

4.4 Extension to Damped Case

Friction (or damping) is a factor that suppresses the movement of oscillators in a real-world environment. The motion of the damped system is gradually reduced due to the presence of a dissipative function $\mathcal{D}$. More precisely, we consider that the dissipative force is proportional to the velocity components of all the bodies of the system. As for the studied two-mass oscillator, we have

$$\mathcal{D}(\dot{\mathbf{s}}) = \frac{1}{2} b_1 \dot{s}_1^2 + \frac{1}{2} b_2 \dot{s}_2^2, \tag{60}$$

where positive constants b_1, b_2 are damping coefficients. Similarly to steps (18)–(21) we change the saddle point system (22) to

$$\begin{bmatrix} \mathbf{M} & \mathbf{G}^{\mathrm{T}} \\ \mathbf{G} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{s}} \\ -\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{V}\mathbf{s} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{D}\dot{\mathbf{s}} \\ 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{D} = \begin{bmatrix} -b_1 & 0 \\ 0 & -b_2 \end{bmatrix}. \tag{61}$$

In the similar way we modify (31) and apply numerical approach from Subsection 3.2.

In what follows we extend the intentional control from Subsection 4.3 to the damped case with $b_1 = 1$ and $b_2 = 0.5$ under the same relevant periods of given phases and data (47). Fig. 6 captures the so-called underdamped system whose components oscillate through the zero equilibrium position.

Remark. The behavior of the system is essentially determined by the mass, stiffness and damping parameters, resulting to the classification of the system as overdamped (with exponential decay), critically damped (moving as fast as possible to equilibrium) or underdamped. The detailed numerical analysis of such a system and estimation of the critical damping ratio are beyond the scope of this study and is left for future research.

Technically speaking, the complex quarter car model contains the spring force, damping force and friction element. The friction element can change the number of degrees of freedom and is modelled by the combination of static friction (occurring when the relative velocity of sprung/unsprung masses is zero) and dynamic friction (representing relative movement). Static friction can lock relative movement (i.e., cause ankylosis) until external forces exceed the friction capacity of the static friction in this element. Then the relative movement is activated again. This phenomenon can be observed especially by gentle mechanisms, such as observation equipment carriers, etc.

The presented methodology can be used for systems of any complexity, as the main idea is to identify the regular part of the system matrix. This part enables solving the system dynamics in any situation; the remaining part has a meaning of the kinematic constraints. The main value of the approach is done in the correct modeling of the friction when the locked kinematic joint is in this approach described as a restraint in the equations of motion. The friction simulation is not based on some alternative approach, such as adding a high-stiffness spring and damper. The approach with additional force elements is still frequently used in the multibody simulation systems. The presented approach is suitable even for the simulation of the singular cases of mechanical systems.

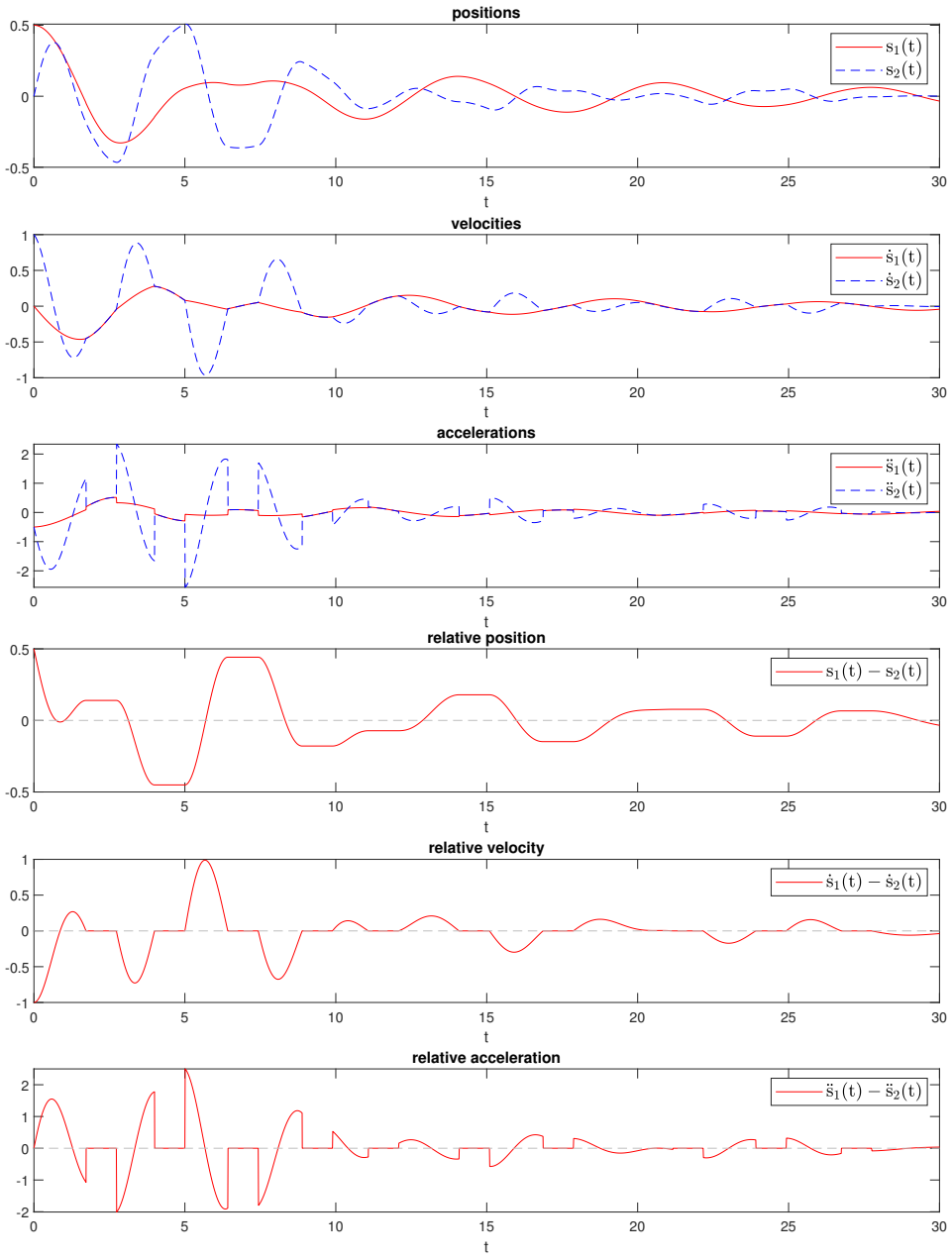

Fig. 6 *Damping: Position, velocity and acceleration components of the damped two-mass oscillator (upper) and the corresponding differences of these components (bottom).*

5. Conclusion

The aim of the paper is to present an advanced methodological concept to treat with mechanical systems subjected to switchable linear constraints, describing mechanism with flexible structures. The cornerstone is the choice of an appropriate coordinate system that allows the constraint to be represented as a linear function with respect to the position components. The studied problem is described using the Lagrangian formalism, leading to DAEs represented by a saddle point problem. From the mathematical point of view, the subsequent elimination of algebraic variables with suitable transformation reformulates the problem to the first order ODE system for unknown position and velocity components. Since saddle point matrices are not invertible in the classical sense due to switchable constraints, a pseudoinverse has to be incorporated. In addition, the time independent scenario simplifies the resulting ODE system to the linear autonomous one.

Special attention is paid to a two-mass oscillator with a switchable constraint to demonstrate functionality. The relevant numerical treatment arises from the time discretization of the differential equations with the aid of Runge-Kutta schemes. The portfolio of numerical experiments produces satisfactory results and illustrates the potency of the presented concept, especially for transient behavior within (damped) intentional control. Moreover, the results obtained reflect the meaningful relevance and can be easily interpreted and subsequently used in a broad spectrum of applications.

Aware that this concept is illustrated under the simplest scenario of a two-mass oscillator, the presented methodology is generally applicable in the computer simulations of the constrained mechanical systems in case of a variable number of degrees of freedom due to blocking or releasing of some kinematic joint, usually indicated as ankylosis. This situation is typical for kinematic joints with friction, where kinematic and static friction coefficients are considered. The systems with the change of structure can be effectively simulated in the case of active systems, where the control system and active elements influence the number of degrees of freedom. The active force elements are integrated into the joints and control the moveability of the mechanism. These situations appear in robotics frequently. The approach helps to solve even the singular cases in the mechanics. The methodology enables the lock of the (unsure) joint, lets the numerical integration make a few steps, and releases the joint just after the singular position when the mechanism continues its movement. Definitely, there is an open possibility to utilize the methodology in applied mechanics, biomechanics, production technology, automotive, and many other disciplines, where mechanical systems play an essential role in the simulations and in the design of digital twins.

Last but not least, let us add from a relevant point of view that the chosen exact approach, in contrast to the currently frequently used modeling using various types of deep neural networks, see e.g., [19,26], gives priority to clear interpretation, stability, exact (and discrete) time solutions and their derivatives (e.g., velocities or accelerations) together with subsequent post-processing in a large number of technical applications, including advanced optimization strategies [15] or systems with nonlinearities/singularities [23].

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