FLOPPY LOGIC – INSTRUCTIONS FOR USE

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tutorial

Abstract: This article provides a simple and practical tutorial on how to use floppy logic. The floppy logic is a method suitable for systems control and description. It preserves the simplicity of the fuzzy logic and the accuracy of the probability theory. The floppy logic allows to work consistently and simultaneously with data in the form of exact numbers, probability distributions and fuzzy sets.

Key words: *floppy logic, floppy sets, fuzzy logic, fuzzy sets, probability*

Received: September 17, 2018 **DOI:** 10.14311/NNW.2018.28.026
Revised and accepted: October 26, 2018

1. Introduction

Floppy logic was introduced in paper [14]. It is a many-valued logic in which the truth values of variables are represented by real numbers between 0 and 1.

This theory is very similar to fuzzy logic [4, 18, 19, 20], but there is an important difference: while variables are described by fuzzy sets in fuzzy logic, in floppy logic they are described by floppy sets. The floppy set is a normal set of primary fuzzy sets.

This difference causes that floppy logic has several advantages which fuzzy logic does not have:

1. Set and logic operations are unambiguous. There are not many triangular norms and co-norms as in fuzzy logic [5, 6, 11].
2. Floppy logic is a model of the Kolmogorov probability theory [9], which means we can use all probabilistic constructions and tools.
3. We can work with exact numbers, probability distributions and fuzzy sets together.
4. All the statements, which are equivalent in standard bivalent logic, are equivalent in floppy logic too.
5. In floppy logic we can use rules of the following type: If A, then $B1$ (with probability of 70 %) or $B2$ (with probability of 30 %).

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The purpose of this article is to teach the reader how to apply this useful theory in practice. Our basic problem, which we will resolve here, has the following structure: Fuzzification of input data, application of system rules, defuzzification of results.

A division of our text has a similar structure:

Section 2 is a brief overview of the most important terms of floppy logic.

In Section 3 we will discuss how to describe the system by fuzzy sets. In this section we will answer the following questions:

- What properties does the primary fuzzy sets system have to be usable for floppy logic?
- How can we modify the primary fuzzy sets system for it to be usable for floppy logic?
- How to create a system of joint primary fuzzy sets from marginal primary fuzzy sets?
- Which primary fuzzy sets describe the system best?

In Section 4 we put the floppy logic assumptions and explain how to fuzzify input variables. We will show it for exact values, probability distributions, floppy sets, intervals and exact sets.

In Section 5 we explain how floppy logic works with logical propositions. Furthermore, we show how to replace IF–THEN rules by conditional probabilities.

In Section 6 we explain how to defuzzify results.

Then Conclusion and Curriculum Vitae follow.

2. The most important terms

Floppy logic is a model of the probability theory. We have used this fact and therefore the formulas that occur in this text are based on standard formulas of the probability theory. For good understanding, it is important to translate the terms of fuzzy and floppy logic into the language of the probability theory.

Primary fuzzy sets Primary fuzzy sets A_i describe some variable A. Single-element set $\{A_i\}$ is an elementary event in probability theory sense.

Membership function of primary fuzzy set Membership function μ_{A_i} of primary fuzzy set A_i is from 0 to 1. The sum of membership functions of all primary fuzzy sets must be 1 everywhere.

(Basic) floppy set All floppy sets used in this paper are the basic floppy sets introduced in [14]. Basic floppy set is a set of primary fuzzy sets. Basic floppy set is an event in probability theory sense. Floppy sets will be denoted by bold symbols.

Membership function of floppy set Membership function $\boldsymbol{\mu}_{\mathbf{B}}$ of floppy set $\mathbf{B}$ is the sum of membership functions of $A_i \in \mathbf{B}$. Membership function $\boldsymbol{\mu}_{\mathbf{B}}$ can be understood as conditional probability $R(\mathbf{B}|x)$. This conditional probability can be often interpreted as the probability that an expert will say that event **B** occurs if the exact value of variable A is x. Membership functions of floppy sets will be denoted by bold symbol $\boldsymbol{\mu}$.

Probability measure R The floppy logic assumes that there is some probability measure P and introduces a new, more general probability measure. This new probability measure will be marked by the symbol R.

3. System description using fuzzy sets

We have a variable A and we want to describe it with the help of a family of fuzzy sets A_1, A_2, … A_n. This means that for every x from the domain X of the variable A we have to determine membership functions $\mu_1(x)$, $\mu_2(x)$, … $\mu_n(x)$. The value $\mu_k(x)$ means the degree of membership of element x to set A_k. This degree of membership must be in the interval $\langle 0, 1\rangle$.

In the floppy logic it is important, that the membership functions meet these two rules:

1. For all $x \in X$ is the sum $\sum_{k=1}^{n} \mu_k(x) = 1$.
2. We choose the sets A_k so that no $x \in X$ can belong to two different A_k, A_m together.

Especially the second rule needs an explanation. In the floppy logic the membership function $\mu_k(x)$ can be interpreted as the probability that an expert determines that variable A has the property A_k, assuming the accurate value of A is x. Thus the second rule says that the primary fuzzy sets A_k have to be chosen so that no expert can say that the variable A has properties A_k and A_m together.

If our system of primary fuzzy sets does not meet some of these two rules, a little modification can often help. If the sum of the membership functions is less than 1, we can add one or more new fuzzy sets. If the sum of the membership functions is higher than 1 or the second rule is not fulfilled, we can substitute two or more fuzzy sets by their combinations.

Now we will show three examples of how to obtain a system of primary fuzzy sets that meets both required rules. The third example also answers the question how to construct fuzzy sets that best match the described reality.

Example 1 – One-dimensional case

Variable A is a temperature of water. To describe the temperature, we have chosen the following properties / fuzzy sets: A_1 – Unhealthily cold, A_3 – Pleasant, A_4 – Warm. Its membership function are shown in Fig. 1(a).

First, we add a fuzzy set A_2 – Healthily cold – so that the sum is nowhere less than one. See Fig. 1(b).

If the sum is for some x higher than one and for this x there are two nonzero fuzzy sets A_i and A_j, we replace fuzzy sets A_i and A_j with new fuzzy sets "Only A_i", "Only A_j" and "A_i and A_j".

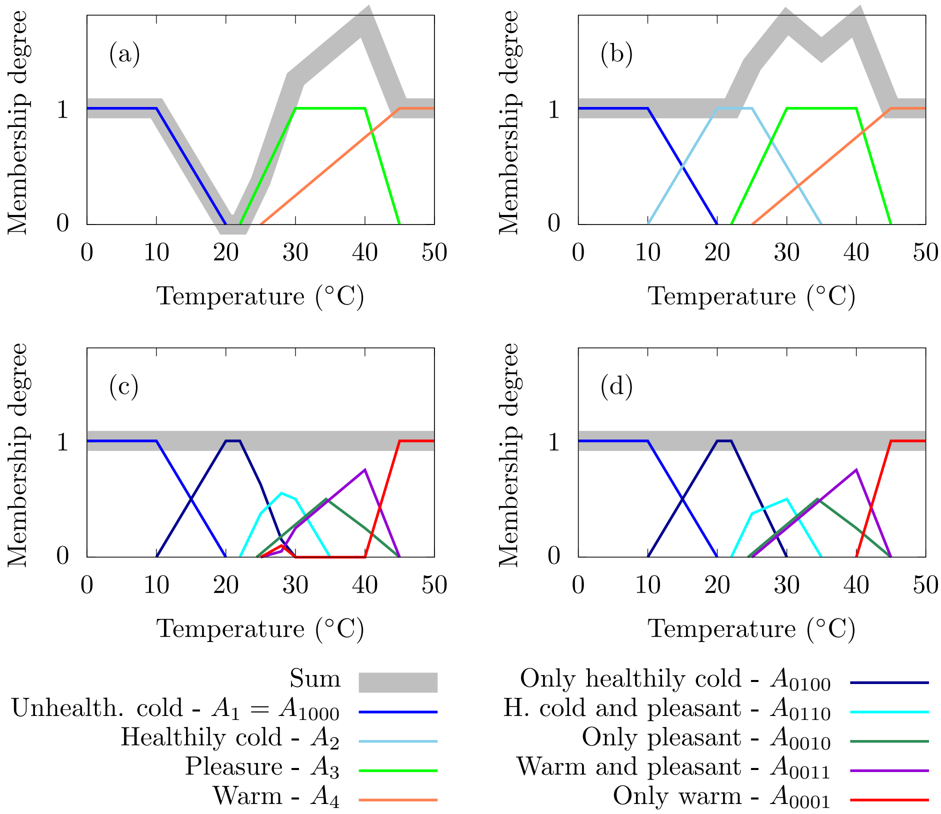

Fig. 1 *Entering of the membership functions.*

The fuzzy sets "Only A_i" will be denoted as A with index with zeros and 1 on i-th position. The fuzzy sets "A_i and A_j" will have ones on i-th and j-th positions.

For simplicity, the membership functions will be denoted in the same way as the fuzzy sets.

For example for fuzzy sets A_3 and A_4 it must be fulfilled:

$$\begin{aligned} A_{0010} + A_{0001} + A_{0011} &= 1, \\ A_{0010} + A_{0011} &= A_3, \\ A_{0001} + A_{0011} &= A_4. \end{aligned}$$

So that:

$$\begin{aligned} A_{0011} &= A_3 + A_4 - 1, \\ A_{0010} &= A_3 - A_{0011}, \\ A_{0001} &= A_4 - A_{0011}. \end{aligned}$$

A membership degree of fuzzy set "A_i and A_j" is the sum minus 1 and we have to subtract the same value from A_i and A_j to get a membership functions of "Only A_i", "Only A_j".

The problem arises if for some x there are more non-zero membership functions. This case occurs for $x \in (25, 35)$.

Fuzzy sets A_2, A_3 and A_4 can be replaced with fuzzy sets A_{0100}, A_{0010}, A_{0001}, A_{0110}, A_{0101}, A_{0011} and A_{0111}. We suppose that no expert determines that the water is cold and warm together. Therefore we eliminate sets A_{0101} and A_{0111}.

It must be fulfilled:

$$\begin{aligned} A_{0100} + A_{0010} + A_{0001} + A_{0110} + A_{0011} &= 1, \\ A_{0100} + A_{0110} &= A_2, \\ A_{0010} + A_{0110} + A_{0011} &= A_3, \\ A_{0001} + A_{0011} &= A_4. \end{aligned}$$

For $x \in \langle 30, 35)$ is the A_3 membership function 1. Thus the membership functions can be calculated unambiguously.

At interval $(25, 30)$, there are many solutions. For example, let us decide that for our mathematical model the best choice is $A_{0110}\,(28) = 0.55$ and $A_{0011}\,(28) = 0.05$. This solution is illustrated in Fig. 1(c).

But this solution is a bit strange. The fuzzy set "Only warm" is non-zero at interval $(25, 30)$ and then at interval $(40, 50)$. If we want a nicer solution, we can divide the fuzzy set "Only warm" into two fuzzy sets.

Another option is to select the values on interval $(25, 30)$ to be the set "Only warm" zero here. This solution is illustrated in Fig. 1(d).

Example 2 – Two-dimensional case – marginal fuzzy sets

Variable T is a temperature of air. We have chosen the following properties / fuzzy sets to describe it: T_1 – Cold, T_2 – Tepid, T_3 – Warm.

Variable P is a pressure of air. We have chosen the following properties / fuzzy sets to describe it: P_1 – Low, P_2 – Normal, P_3 – High.

These fuzzy sets meet the two rules above.

We would like to describe the state of the air by temperature and pressure together. Therefore, the Cartesian product of temperature and pressure will be the domain of the membership functions of the fuzzy sets. Set T_1 will be understood as "it is cold and pressure is arbitrary". Similarly, other sets can be understood. These marginal fuzzy sets are shown in Fig. 2(a) and (b).

The sum of all the membership function of T_i and P_j is 2 everywhere. So we have to introduce fuzzy sets "T_i and P_j". These fuzzy sets will be denoted $A_{i,j}$.

Fuzzy sets "Only T_i" and "Only P_j" will be denoted $A_{i,0}$ and $A_{0,j}$.

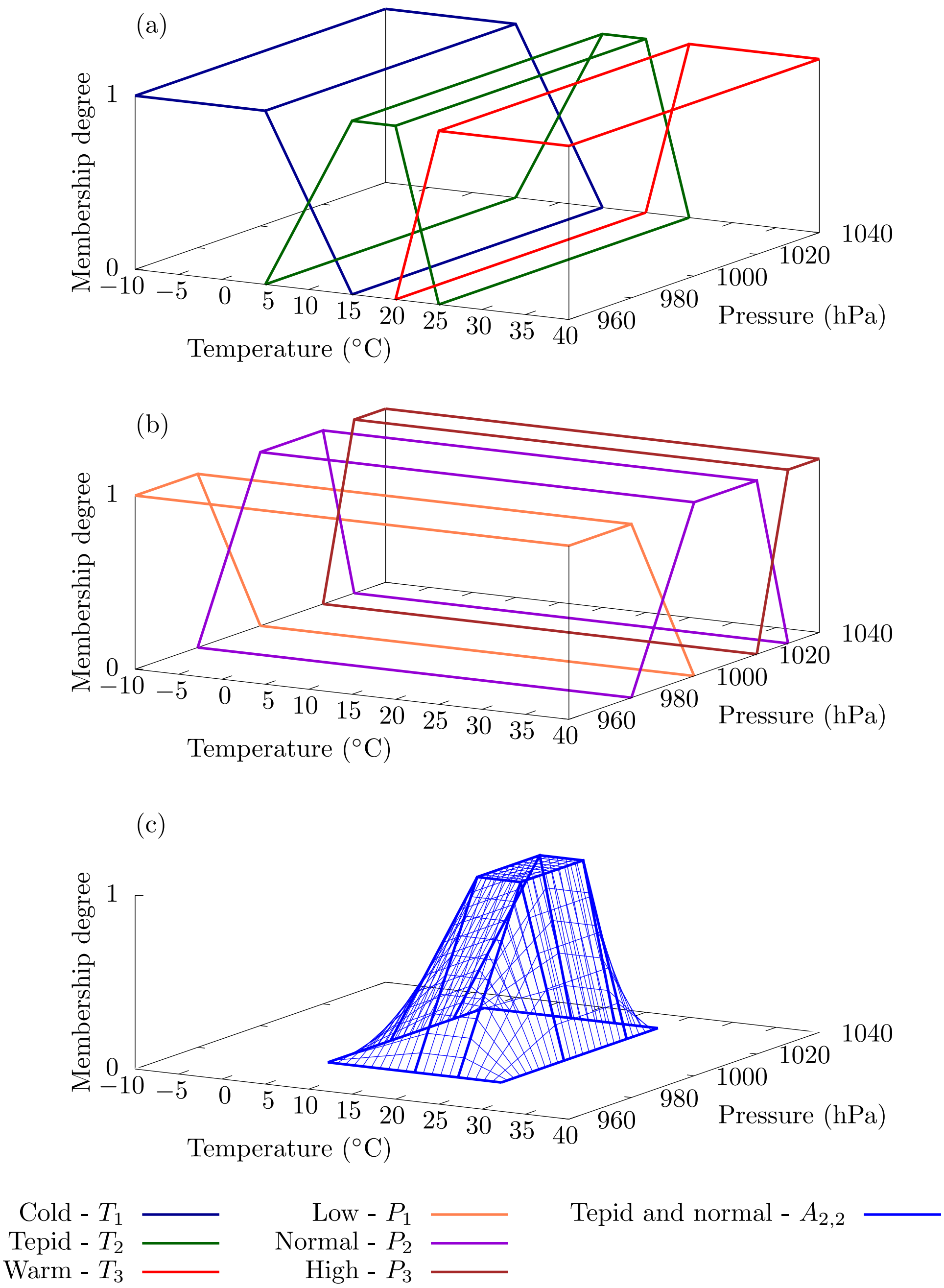

Fig. 2 *Marginal and joint membership functions.*

We solve the following system of equations:

$$\begin{aligned}
\sum_i A_{i,0} + \sum_j A_{0,j} + \sum_i \sum_j A_{i,j} &= 1, \\
A_{i,0} + \sum_j A_{i,j} &= T_i, \\
A_{0,j} + \sum_i A_{i,j} &= P_j, \\
\sum_i T_i &= 1, \\
\sum_j P_j &= 1.
\end{aligned}$$

By solving this system, we find that the membership functions of all fuzzy sets "Only T_i" and "Only P_j" must be zero.

The system has an unambiguous solution only for a small number of non-zero fuzzy sets. Larger systems have more solutions.

But one solution, always valid, is very simple:

$$A_{i,j} = T_i \cdot P_j.$$

So, we replace fuzzy sets T_i and P_j with nine fuzzy sets $A_{i,j} = T_i \cdot P_j$.

Fuzzy set "Tepid and normal" obtained in this way is shown in Fig. 2(c).

Multiplication of membership functions of marginal fuzzy sets is of course possible even in more than two-dimensional case.

Example 3 – Two-dimensional case – clusters

This example is inspired by article [2] where the authors describe a fuzzy C-means clustering algorithm. This method was introduced in [1].

We need to divide the points in the plane into three clusters. These clusters are understood as fuzzy sets $A_1, \dots A_3$. Clusters centers are at points $C_1, \dots C_3$. How to make clusters memberships functions which meet the two rules above?

The method can be very simple: Let us assign a positive, non-zero[1] function $g_i(X)$ to each cluster. For example, it can be function $g_i(X) = \frac{1}{d_i}$ where d_i is the distance between points X and C_i.[2] It is a standard choice in C-means method.

Now we calculate cluster membership functions according to the formula:

$$\mu_{A_i}(X) = \frac{g_i(X)}{\sum_j g_j(X)}.$$

All positive, non-zero functions g_i are permissible but which functions match the described reality best? The best choice is density of intensity, as shown in the next part. It can often be calculated as the probability density function of the cluster times the intensity of the cluster.

[1] There must be at least one non-zero function everywhere.

[2] Of course, the singularities must be solved.

For example, our three clusters have three different multivariate normal distributions (see [10]) with mean vectors:

$$\mathbf{m}_1 = \begin{pmatrix} -100 \\ 30 \end{pmatrix}, \qquad \mathbf{m}_2 = \begin{pmatrix} 20 \\ -5 \end{pmatrix}, \qquad \mathbf{m}_3 = \begin{pmatrix} 80 \\ 0 \end{pmatrix},$$

covariance matrices:

$$\mathbf{M}_1 = \begin{pmatrix} 1600 & -800 \\ -800 & 1600 \end{pmatrix}, \qquad \mathbf{M}_2 = \begin{pmatrix} 200 & 0 \\ 0 & 200 \end{pmatrix}, \qquad \mathbf{M}_3 = \begin{pmatrix} 300 & 100 \\ 100 & 300 \end{pmatrix}$$

and with clusters intensities of 70, 50 and 80 points per minute.

These three membership functions are shown in Fig. 3(a), (b), (c). But they are a bit strange. A large membership degree is also at places very distant from centers.

How to improve it? We can be inspired by article [17], where authors take into consideration not only clusters but also backgrounds. We add a new fuzzy set A_4 – "Backgrounds". Its function $g_4(X)$ is constant everywhere. For example intensity of this fuzzy set is 30 points per minute in rectangle 300×200. Therefore $g_4 = \frac{30}{300 \cdot 200}$.

The membership functions of all A_i are computed as it is shown above. The results can be seen in Fig. 3(d), (e), (f), (g). These functions typify the degree of cluster membership much better.

4. Fuzzification

4.1 Assumptions of floppy logic

Now we have system $\mathbf{S}$ of primary fuzzy sets which describe a variable A and which meet the two rules from the previous chapter. The domain of the primary fuzzy sets is X. The subsets of $\mathbf{S}$ will be called basic floppy sets. The task of this section is to calculate the probabilities of basic floppy sets.

We introduce the membership function $\boldsymbol{\mu}_{\mathbf{B}}(x)$ of a basic floppy set $\mathbf{B} = \{A_i, A_j, A_k, \ldots\}$ as the sum of the membership functions of its elements.

Let us suppose that system $\mathbf{S}$ meets the following five assumptions:

Assumption 1. $\mathbf{S}$ *is a finite or countable set.*

Assumption 2. *Membership functions of fuzzy sets $A_i \in \mathbf{S}$ assume values from the interval $\langle 0, 1 \rangle$.*

Assumption 3. *The sum of all the membership functions of fuzzy sets $A_i \in \mathbf{S}$ is equal to one everywhere:*

$$\forall x \in X : \sum_{A_i \in \mathbf{S}} \mu_{A_i}(x) = 1.$$

Assumption 4. *A measure space $(X, \mathcal{A}, P)$ is defined on set X where $\mathcal{A}$ is a σ-algebra on X and P is a probability measure.*

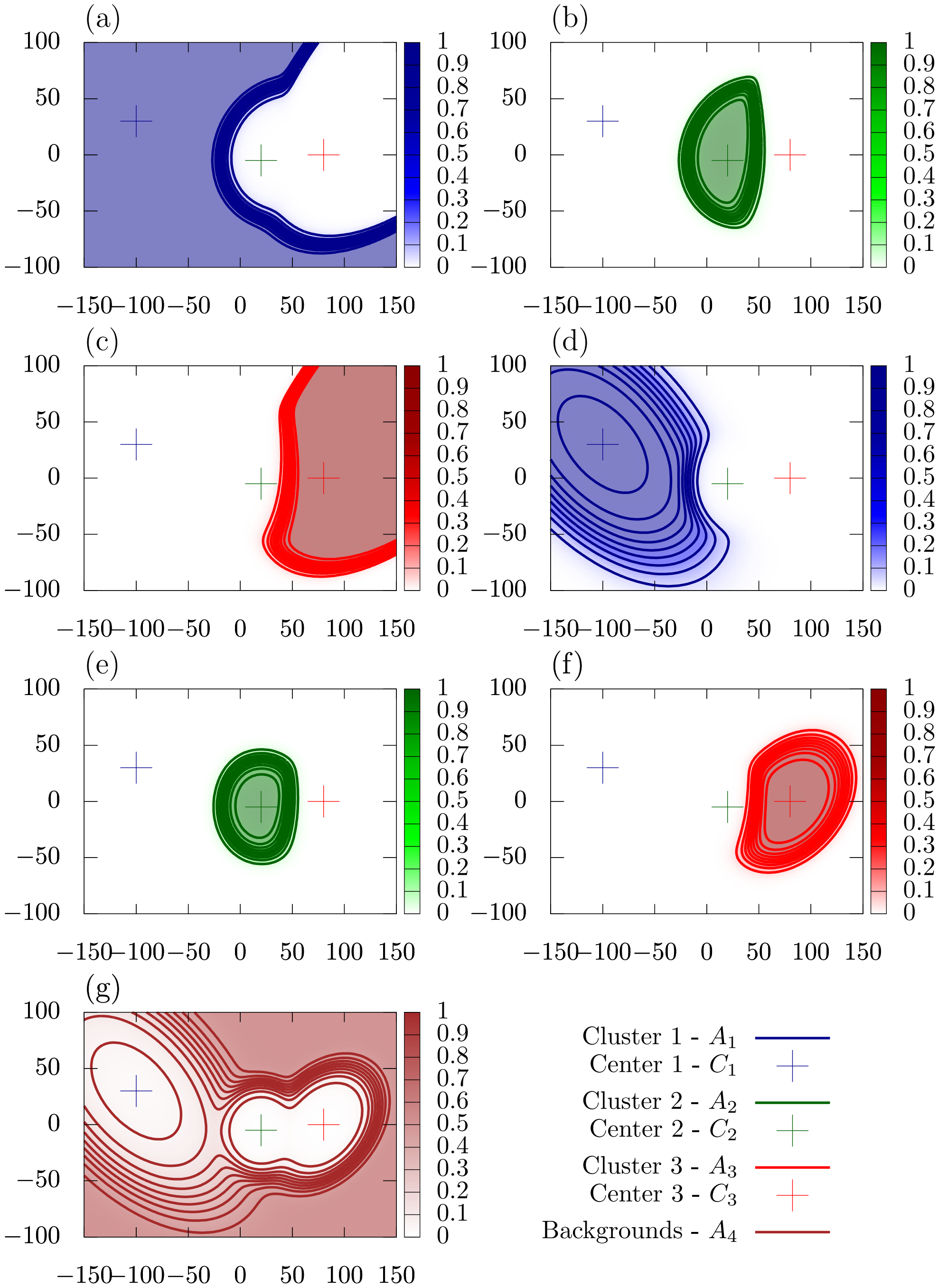

Fig. 3 *Clusters membership functions.*

Assumption 5. *All the membership functions μ_{A_i} of fuzzy sets $A_i \in \mathbf{S}$ are measurable on the sets $X_i \in \mathcal{A}$ with respect to measure P.*

Then the membership function of floppy set $\mathbf{B}$ can be understood as conditional probability $R(\mathbf{B}|x)$ where $x \in X$.[3]

How to calculate probability $R(\mathbf{B})$, if we have some data?

4.2 Exact numbers

We know that the exact value of variable A is x_0.

The probability of floppy set $\mathbf{B}$ will be calculated according to the formula:

$$R(\mathbf{B}) = R(\mathbf{B}|x_0) \cdot R(x_0) = \boldsymbol{\mu}_{\mathbf{B}}(x_0).$$

Example: What is the probability that a point with coordinates $[10, 40]$ belongs to cluster A_2?

$$R(\{A_2\}) = \boldsymbol{\mu}_{\{A_2\}}([10, 40]) = \frac{g_2([10, 40])}{\sum\limits_{j=1}^{4} g_j([10, 40])},$$

where

$$\boldsymbol{\mu}_{\{A_2\}}(x) = \mu_{A_2}(x).$$

The probability is equal to the ratio of intensity density, which is reasonable. This is the reason why the choice of fuzzy sets shown in Example 3 is the best.

4.3 Probability distribution

We know a probability distribution of variable A. It can be the probability distribution supposed in Assumptions 4 and 5.

The probability of floppy set $\mathbf{B}$ can be calculated according to the formula:

$$R(\mathbf{B}) = \int_X \boldsymbol{\mu}_{\mathbf{B}}(x)\, dP,$$

where the integral is the Lebesgue integral.

If A is a discrete random variable then we can use a simpler formula:

$$R(\mathbf{B}) = \sum_{x_i \in X} \boldsymbol{\mu}_{\mathbf{B}}(x_i) \cdot P(x_i),$$

where $P(x_i)$ is the known probability function.

If A is a continuous random variable then we can use a formula:

$$R(\mathbf{B}) = \int_X \boldsymbol{\mu}_{\mathbf{B}}(x) \cdot f(x)\, dx,$$

where the integral is the Riemann integral and $f(x)$ is the known probability density.

[3]We write "R" instead of "P" because P is the probability measure on X supposed in Assumptions 4 and 5. R is a probability measure on a more general set.

Example: We know joint probability distribution of temperature T and pressure P of air. What is the probability that the pressure is high?

$$R\left(\{A_{1,3},A_{2,3},A_{3,3}\}\right)=\int_{P=0}^{\infty}\int_{T=-\infty}^{\infty}\boldsymbol{\mu}_{\{A_{1,3},A_{2,3},A_{3,3}\}}\left(T,P\right)\cdot f\left(T,P\right)\,dT\,dP,$$

where

$$\boldsymbol{\mu}_{\{A_{1,3},A_{2,3},A_{3,3}\}}\left(T,P\right)=\mu_{A_{1,3}}\left(T,P\right)+\mu_{A_{2,3}}\left(T,P\right)+\mu_{A_{3,3}}\left(T,P\right).$$

4.4 Floppy set and exact value

We know that variable A is described by floppy set $\mathbf{C}$ and the exact value of variable A is x_0.

The probability of floppy set $\mathbf{B}$ will be calculated according to the formula:

$$R\left(\mathbf{B}\right)=R\left(\mathbf{B}|\mathbf{C},\mathbf{x_0}\right)=\frac{R\left(\mathbf{B}\cap\mathbf{C}|\mathbf{x_0}\right)}{R\left(\mathbf{C}|\mathbf{x_0}\right)}=\frac{\boldsymbol{\mu}_{\mathbf{B}\cap C}\left(x_0\right)}{\boldsymbol{\mu}_{\mathbf{C}}\left(x_0\right)}.$$

Example: A point has coordinates $[0,50]$. We know that the point belongs to cluster A_2 or to cluster A_3. What is the probability that the point belongs to cluster A_3?

$$\begin{aligned}R\left(\{A_3\}\right)&=\frac{\boldsymbol{\mu}_{\{A_3\}\cap\{A_2,A_3\}}\left([0,50]\right)}{\boldsymbol{\mu}_{\{A_2,A_3\}}\left([0,50]\right)}=\frac{\boldsymbol{\mu}_{\{A_3\}}\left([0,50]\right)}{\boldsymbol{\mu}_{\{A_2,A_3\}}\left([0,50]\right)}=\\&=\frac{\frac{g_3([0,50])}{\sum\limits_{j=1}^{4}g_j([0,50])}}{\frac{g_2([0,50])+g_3([0,50])}{\sum\limits_{j=1}^{4}g_j([0,50])}}=\frac{g_3\left([0,50]\right)}{g_2\left([0,50]\right)+g_3\left([0,50]\right)}.\end{aligned}$$

4.5 Floppy set and probability distribution

We know that variable A is described by floppy set $\mathbf{C}$ and variable A has a probability distribution. Of course, we use the best estimate of the probability distribution we know.

The probability of floppy set $\mathbf{B}$ can be calculated according to the formula:

$$R\left(\mathbf{B}\right)=R\left(\mathbf{B}|\mathbf{C}\right)=\frac{R\left(\mathbf{B}\cap\mathbf{C}\right)}{R\left(\mathbf{C}\right)}=\frac{\int_X\boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}\left(x\right)dP}{\int_X\boldsymbol{\mu}_{\mathbf{C}}\left(x\right)dP},$$

where the integrals are the Lebesgue integrals.

If A is a discrete random variable then we can use a formula:

$$R\left(\mathbf{B}\right)=\frac{\sum_{x_i\in X}\boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}\left(x_i\right)\cdot P\left(x_i\right)}{\sum_{x_i\in X}\boldsymbol{\mu}_{\mathbf{C}}\left(x_i\right)\cdot P\left(x_i\right)}.$$

If A is a continuous random variable then we use a formula:

$$R\left(\mathbf{B}\right)=\frac{\int_X\boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}\left(x\right)\cdot f\left(x\right)dx}{\int_X\boldsymbol{\mu}_{\mathbf{C}}\left(x\right)\cdot f\left(x\right)dx},$$

where the integrals are the Riemann integrals.

Example: We know, that the water is cold. The probability density function of temperature is $f(x)$. What is the probability that the water is pleasant?

$$R(\{A_{0110}, A_{0010}, A_{0011}\}) =$$

$$= \frac{\int_X \boldsymbol{\mu}_{\{A_{0110}, A_{0010}, A_{0011}\} \cap \{A_{1000}, A_{0100}, A_{0110}\}}(x) \cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{1000}, A_{0100}, A_{0110}\}}(x) \cdot f(x)\,dx} =$$
$$= \frac{\int_X \boldsymbol{\mu}_{\{A_{0110}\}}(x) \cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{1000}, A_{0100}, A_{0110}\}}(x) \cdot f(x)\,dx}.$$

4.6 Interval and probability density function

We know that the value of continuous random variable A lies in the interval $\langle a, b\rangle$ and the probability density function is $f(x)$.

The probability of floppy set $\mathbf{B}$ can be calculated according to the formula:

$$R(\mathbf{B}) = R(\mathbf{B}|\langle a, b\rangle) = \frac{R(\mathbf{B} \cap \langle a, b\rangle)}{R(\langle a, b\rangle)} = \frac{\int_a^b \boldsymbol{\mu}_{\mathbf{B}}(x) \cdot f(x)\,dx}{\int_a^b f(x)\,dx}.$$

Example: The temperature of water is in interval $\langle 20, 40\rangle$ degree Celsius. What is the probability that the water is pleasant?

$$R(\{A_{0110}, A_{0010}, A_{0011}\}) = \frac{\int_{20}^{40} \boldsymbol{\mu}_{\{A_{0110}, A_{0010}, A_{0011}\}}(x) \cdot f(x)\,dx}{\int_{20}^{40} f(x)\,dx}.$$

4.7 Floppy set, interval and probability density function

We know that the continuous random variable A is described by floppy set $\mathbf{C}$, its exact value lies within the interval $\langle a, b\rangle$ and the probability density function is $f(x)$.

The probability of floppy set $\mathbf{B}$ will be calculated according to the formula:

$$\begin{aligned} R(\mathbf{B}) = R(\mathbf{B}|\mathbf{C}, \langle a, b\rangle) &= \frac{R(\mathbf{B} \cap \mathbf{C}|\langle \mathbf{a}, \mathbf{b}\rangle)}{R(\mathbf{C}|\langle \mathbf{a}, \mathbf{b}\rangle)} = \\ &= \frac{\frac{\int_a^b \boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}(x) \cdot f(x)dx}{\int_a^b f(x)dx}}{\frac{\int_a^b \boldsymbol{\mu}_{\mathbf{C}}(x) \cdot f(x)dx}{\int_a^b f(x)dx}} = \\ &= \frac{\int_a^b \boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}(x) \cdot f(x)\,dx}{\int_a^b \boldsymbol{\mu}_{\mathbf{C}}(x) \cdot f(x)\,dx}. \end{aligned}$$

Example: Water is not warm and the temperature of water is over 15 degree Celsius. The probability density function is $f(x)$. What is the probability that the water is pleasant?

$$R\left(\{A_{0110},A_{0010}\}\right) = \frac{\int_{15}^{\infty}\boldsymbol{\mu}_{\{A_{0110},A_{0010}\}\cap\{A_{1000},A_{0100},A_{0110},A_{0010}\}}(x)\cdot f(x)\,dx}{\int_{15}^{\infty}\boldsymbol{\mu}_{\{A_{1000},A_{0100},A_{0110},A_{0010}\}}(x)\cdot f(x)\,dx} = \frac{\int_{15}^{\infty}\boldsymbol{\mu}_{\{A_{0110},A_{0010}\}}(x)\cdot f(x)\,dx}{\int_{15}^{\infty}\boldsymbol{\mu}_{\{A_{1000},A_{0100},A_{0110},A_{0010}\}}(x)\cdot f(x)\,dx}.$$

4.8 Floppy set, subset of X and probability density function

A variable A is described by floppy set $\mathbf{C}$ and has a probability distribution. The exact value of A lies in set Y which is a subset of X.

The probability of floppy set $\mathbf{B}$ can be calculated according to the formula:

$$R(\mathbf{B}) = R(\mathbf{B}|\mathbf{C},Y) = \frac{\int_Y \boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}(x)\,dP}{\int_Y \boldsymbol{\mu}_{\mathbf{C}}(x)\,dP},$$

where the integrals are the Lebesgue integrals.

If A is a discrete random variable then we can use the formula:

$$R(\mathbf{B}) = \frac{\sum_{x_i\in Y}\boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}(x_i)\cdot P(x_i)}{\sum_{x_i\in Y}\boldsymbol{\mu}_{\mathbf{C}}(x_i)\cdot P(x_i)}.$$

If A is a continuous random variable then we use the formula:

$$R(\mathbf{B}) = \frac{\int_Y \boldsymbol{\mu}_{\mathbf{B}\cap\mathbf{C}}(x)\cdot f(x)\,dx}{\int_Y \boldsymbol{\mu}_{\mathbf{C}}(x)\cdot f(x)\,dx},$$

where the integrals are the Riemann integrals.

Example: It is warm. Temperature of air T is below 32 degree Celsius and pressure P is over 952 hPa. $\frac{P}{T} \leq 34\,\mathrm{hPa}/^{\circ}\mathrm{C}$. The joint probability density function is $f(T,P)$. What is the probability that the pressure is low?

$$\begin{aligned} R\left(\{A_{1,1},A_{2,1},A_{3,1}\}\right) &= \frac{\int_Y \boldsymbol{\mu}_{\{A_{1,1},A_{2,1},A_{3,1}\}\cap\{A_{3,1},A_{3,2},A_{3,3}\}}(T,P)\cdot f(T,P)\,dx}{\int_Y \boldsymbol{\mu}_{\{A_{3,1},A_{3,2},A_{3,3}\}}(T,P)\cdot f(T,P)\,dx} = \\ &= \frac{\int_Y \boldsymbol{\mu}_{\{A_{3,1}\}}(T,P)\cdot f(T,P)\,dx}{\int_Y \boldsymbol{\mu}_{\{A_{3,1},A_{3,2},A_{3,3}\}}(T,P)\cdot f(T,P)\,dx} = \\ &= \frac{\int_{T=28}^{32}\int_{P=952}^{34\cdot T}\boldsymbol{\mu}_{\{A_{3,1}\}}(T,P)\cdot f(T,P)\;dP\,dT}{\int_{T=28}^{32}\int_{P=952}^{34\cdot T}\boldsymbol{\mu}_{\{A_{3,1},A_{3,2},A_{3,3}\}}(T,P)\cdot f(T,P)\;dP\,dT}. \end{aligned}$$

5. Rules of system

5.1 Logical propositions

A system can be usually described by a set of rules of the structure: $\alpha \Rightarrow \beta$ where α and β are logical propositions.

We modify these propositions α and β for use in floppy logic:

1. We replace these propositions with other equivalent ones that only contain negations and logical connectives "and" and "or". It is not important which of the equivalent propositions we choose. They all lead to the same result. In this sense, propositions equivalent in standard logic are also equivalent in floppy logic.

2. We replace the negations by the set complements, the logical connectives "and" and "or" by the set operators "intersection" and "union", the elementary propositions by the relevant floppy sets.

3. We express the result floppy set.

Example: We use the first example from Section 3.

We have a logical proposition, e.g.: "The water is warm if and only if the water is cold." We can write:

$$W \Leftrightarrow C.$$

We replace the proposition by:

$$(W \wedge C) \vee (\neg W \wedge \neg C)\,.$$

We replace the elementary propositions by the relevant floppy sets (See Fig. 1.) and the logical connectives by the set operators:

$$\begin{aligned}
\Big(\{A_{0011}, A_{0001}\} \cap \{A_{1000}, A_{0100}, A_{0110}\}\Big) \cup \\
\cup \Big(\{A_{0011}, A_{0001}\}^{\mathsf{C}} \cap \{A_{1000}, A_{0100}, A_{0110}\}^{\mathsf{C}}\Big) &= \\
= \Big(\{A_{0011}, A_{0001}\} \cap \{A_{1000}, A_{0100}, A_{0110}\}\Big) \cup \\
\cup \Big(\{A_{1000}, A_{0100}, A_{0110}, A_{0010}\} \cap \{A_{0010}, A_{0011}, A_{0001}\}\Big) &= \\
= \emptyset \cup \{A_{0010}\} &= \{A_{0010}\}\,.
\end{aligned}$$

We could choose another proposition equivalent to $W \Leftrightarrow C$. For example:

$$(W \Rightarrow C) \wedge (C \Rightarrow W)\,.$$

We edit:

$$(C \vee \neg W) \wedge (W \vee \neg C)\,.$$

We replace the propositions and logical connectives:

$$\begin{aligned}
\Big(\{A_{1000}, A_{0100}, A_{0110}\} \cup \{A_{0011}, A_{0001}\}^{\mathsf{C}}\Big) \cap \\
\cap \Big(\{A_{0011}, A_{0001}\} \cup \{A_{1000}, A_{0100}, A_{0110}\}^{\mathsf{C}}\Big) &= \\
\Big(\{A_{1000}, A_{0100}, A_{0110}\} \cup \{A_{1000}, A_{0100}, A_{0110}, A_{0010}\}\Big) \cap \\
\cap \Big(\{A_{0011}, A_{0001}\} \cup \{A_{0010}, A_{0011}, A_{0001}\}\Big) &= \\
\{A_{1000}, A_{0100}, A_{0110}, A_{0010}\} \cap \{A_{0010}, A_{0011}, A_{0001}\} &= \{A_{0010}\}\,.
\end{aligned}$$

5.2 Implications

In Section 4 we calculated the probabilities of the input floppy sets. The goal of this part is to calculate the probabilities of the output floppy sets.

First we must realize that there are two probabilities of output floppy sets:

1. The a priori probabilities which can be computed without a knowledge of the input floppy sets from the a priori probability distribution this way:

$$R(\mathbf{B}) = \sum_{x_i \in X} \boldsymbol{\mu}_{\mathbf{B}}(x_i) \cdot P(x_i)$$

   in discrete case, or:

$$R(\mathbf{B}) = \int_X \boldsymbol{\mu}_{\mathbf{B}}(x) \cdot f(x)\, dx$$

   in continuous case. The a priori probability distribution is assumed in the assumptions of floppy logic.

2. The posteriori probabilities which will be computed from the probabilities of input floppy sets. These posteriori probabilities will be denoted by index P. For example: $R^P(\mathbf{B})$.

In this section we will calculate the posteriori probabilities of output floppy sets.

For this purpose we will model the rules $\alpha \Rightarrow \beta$ by relevant conditional probabilities:

We will replace propositions α_i and β_j by relevant floppy sets $\mathbf{A}_i$ and $\mathbf{B}_j$. The rule $\alpha_i \Rightarrow \beta_j$ will be replaced by relationship $R(\mathbf{B}_j|\mathbf{A}_i) = 1.$[4]

The posteriori probability of $\mathbf{B}_j$ will be computed by the law of total probability (E.g.[15].):

$$R^P(\mathbf{B}_j) = \sum_i R(\mathbf{B}_j|\mathbf{A}_i) \cdot R(\mathbf{A}_i).$$

It is very practical to write numbers $R(\mathbf{B}_j|\mathbf{A}_i)$ into matrix $\overleftrightarrow{R(\mathbf{B}|\mathbf{A})}$, numbers $R(\mathbf{A}_i)$ to column vector $\overrightarrow{R(\mathbf{A})}$ and numbers $R^P(\mathbf{B}_j)$ to the column vector $\overrightarrow{R^P(\mathbf{B})}$. Then we can write:

$$\overrightarrow{R^P(\mathbf{B})} = \overleftrightarrow{R(\mathbf{B}|\mathbf{A})} \cdot \overrightarrow{R(\mathbf{A})}.$$

Matrix $\overleftrightarrow{R(\mathbf{B}|\mathbf{A})}$ can model propositions of the following type: If A_1, then B_1 in 30 % or B_2 in 70 %. It only must be fulfilled that the sum of values in the each column is 1.

[4]If we insist that the numbers $R(\mathbf{B}_j|\mathbf{A}_i)$ are the conditional probabilities, thus $R(\mathbf{B}_j|\mathbf{A}_i) = \frac{R(\mathbf{B}_j \cap \mathbf{A}_i)}{R(\mathbf{A}_i)}$, then the membership functions of floppy sets $\mathbf{A}_i$ and $\mathbf{B}_j$ must have the same domain. We proceed as in Example 2 in Section 3. But this formal step often can be omitted.

Example: A system can be described by the following rules:

1. If we give the medicine to a patient from "cluster 1", his condition will be better.

2. If we give the medicine to a patient from "cluster 2", his condition will be better in 70 % or the same in 30 % of cases.

3. If we give the medicine to a patient from "cluster 3", his condition will be worse.

4. If we give the medicine to a patient from "background", his condition will be better in 50 %, the same in 30 % or worse in 20 % of cases.

So, we have four input floppy sets (A_1 = cluster 1, A_2 = cluster 2, A_3 = cluster 3, A_4 = background) and three output floppy sets (B_1 = better, B_2 = the same, B_3 = worse). The known probabilities of input floppy sets will be aligned into column vector $\overrightarrow{R\,(\mathbf{A})}$. The unknown probabilities of output floppy sets will be aligned into column vector $\overrightarrow{R^P(\mathbf{B})}$. The matrix of the conditional probabilities will be:

$$\overleftrightarrow{R\,(\mathbf{B}|\mathbf{A})} = \begin{pmatrix} 1 & 0.7 & 0 & 0.5 \\ 0 & 0.3 & 0 & 0.3 \\ 0 & 0 & 1 & 0.2 \end{pmatrix}.$$

We compute the unknown probabilities: $\overrightarrow{R^P(\mathbf{B})} = \overleftrightarrow{R\,(\mathbf{B}|\mathbf{A})} \cdot \overrightarrow{R\,(\mathbf{A})}$.

6. Defuzzification

6.1 Calculation of posteriori probability distribution

In the last section we got the probabilities of output floppy sets. This form of output data can be absolutely satisfactory in some situations. E.g.: It will be sunny in 80 %, cloudy in 15 % and rainy in 5 %.

Sometimes I want to know which floppy set has the highest probability. E.g.: It will be probably sunny.

Sometimes I am interested in some point estimation. For example: It will be 28°C. These point estimations, as mean or median, can be computed from probability distribution.

So, the goal of this section is calculation of posteriori probability distribution from the posteriori probabilities of the output floppy sets. Then the point or interval estimations can be calculated as usual.

The posteriori probability distribution will be denoted by index P.

In order to calculate this probability distribution, we need a system of output floppy sets $\{\mathbf{B}_j\}$, which have two properties:

1. All output floppy sets in this system are pairwise disjoint. So the intersection of two different floppy sets is $\emptyset$.

2. The sum of the probabilities of the output floppy sets from this system is 1.

If these properties are satisfied, the posteriori probability distribution will be computed by the law of total probability and by Bayes' theorem:

$$\begin{aligned} R^P(x_i) &= \sum_j R(x_i|\mathbf{B}_j)\cdot R^P(\mathbf{B}_j) = \\ &= \sum_j \frac{R(\mathbf{B}_j|x_i)\cdot R(x_i)}{R(\mathbf{B}_j)}\cdot R^P(\mathbf{B}_j) = \\ &= \sum_j \frac{\boldsymbol{\mu}_{\mathbf{B}_j}(x_i)\cdot P(x_i)}{\sum_{x_i\in X}\boldsymbol{\mu}_{\mathbf{B}_j}(x_i)\cdot P(x_i)}\cdot R^P(\mathbf{B}_j) \end{aligned}$$

for discrete case, or:

$$\begin{aligned} f^P(x) &= \sum_j f(x|\mathbf{B}_j)\cdot R^P(\mathbf{B}_j) = \\ &= \sum_j \frac{R(\mathbf{B}_j|x)\cdot f(x)}{R(\mathbf{B}_j)}\cdot R^P(\mathbf{B}_j) = \\ &= \sum_j \frac{\boldsymbol{\mu}_{\mathbf{B}_j}(x)\cdot f(x)}{\int_{x\in X}\boldsymbol{\mu}_{\mathbf{B}_j}(x)\cdot f(x)\,dx}\cdot R^P(\mathbf{B}_j) \end{aligned}$$

for continuous case.

6.2 New system of output floppy sets

If the properties on page 488 are not satisfied, you must make a new system of output floppy sets. Ideally the most detailed possible.

Example: The temperature of water is described by three output fuzzy sets: "A_w – warm", "A_t – tepid" and "A_c – cold". These fuzzy sets satisfy the assumptions of floppy logic. We know: The probability that water is not cold, is 80 % and the probability that water is not warm, is 50 %.

Floppy sets $\{A_w, A_t\}$ – "not cold" and $\{A_t, A_c\}$ – "not warm" are not pairwise disjoint. So we must make a new system of output floppy sets. We choose the most detailed possible: $\{A_w\}$, $\{A_t\}$, $\{A_c\}$.

We have to solve the following system of equations:

$$\begin{aligned} 1 &= R^P(\{A_w\}) + R^P(\{A_t\}) + R^P(\{A_c\})\,, \\ 0.8 &= R^P(\{A_w\}) + R^P(\{A_t\})\,, \\ 0.5 &= R^P(\{A_t\}) + R^P(\{A_c\})\,. \end{aligned}$$

We get:

$$\begin{aligned} R^P(\{A_w\}) &= 0.5, \\ R^P(\{A_t\}) &= 0.3, \\ R^P(\{A_c\}) &= 0.2. \end{aligned}$$

In this case, we have obtained unambiguous results for the probabilities we were looking for. In some other cases, a similar system of equations may not have a solution or the solution will not be from the interval $\langle 0, 1\rangle$.

Example: Weather is described by three fuzzy sets: "sunny", "cloudy", "rainy". Water is described as in Example 1 in Section 3. A system is described by the following rules:

1. If it is sunny, then the water is not cold in 90 % or is cold in 10 % of cases.
2. If it is not rainy, then the water is warm.

It is sunny. What are the probabilities of all single-element floppy sets?
The first rule gives:

$$\begin{aligned} 0.9 &= R^P(\{A_{0010}\}) + R^P(\{A_{0011}\}) + R^P(\{A_{0001}\}) \,, \\ 0.1 &= R^P(\{A_{1000}\}) + R^P(\{A_{0100}\}) + R^P(\{A_{0110}\}) \,. \end{aligned}$$

The second rule gives:

$$1 = R^P(\{A_{0011}\}) + R^P(\{A_{0001}\}) \,.$$

Therefore:

$$R^P(\{A_{0010}\}) = -0.1.$$

So, the system of rules is not consistent and must be changed.

Note that it is not possible to create matrix $\overleftrightarrow{R\,(\mathbf{B}|\mathbf{A})}$ in this example. The first and the second rule give different regulations for the same matrix column.

Sometimes a similar system of equations has several possible solutions. In this case, we can try to bring together the output fuzzy sets which appear together in the system rules. The second good idea is to draw knowledge from a priori probabilities.

Example: The weather is described by three fuzzy sets: "sunny", "cloudy", "rainy". The water is described as in Example 1 in Section 3. The system is described by the following rules:

1. If it is sunny, then the water is not cold in 90 % or is cold in 10 % of cases.
2. If it is cloudy, then the water is warm in 50 % or is not warm in 50 % of cases.
3. If it is rainy, then the water is warm in 10 % or is cold in 40 % of cases.

We determine three output floppy sets:

- cold – $\mathbf{B}_1 = \{A_{1000}, A_{0100}, A_{0110}\}$,
- not cold and not warm – $\mathbf{B}_2 = \{A_{0010}\}$,
- warm – $\mathbf{B}_3 = \{A_{0011}, A_{0001}\}$.

We try to create matrix $\overleftrightarrow{R\,(\mathbf{B}|\mathbf{A})}$:

$$\overleftrightarrow{R\,(\mathbf{B}|\mathbf{A})} = \begin{pmatrix} 0.1 & c & 0.4 \\ a & d & 0.5 \\ b & 0.5 & 0.1 \end{pmatrix} .$$

We know that $a+b=0.9$ and $c+d=0.5$. Numbers a, b, c and d will be estimated from a priori probabilities:

$$\begin{aligned}
a &= R\left(\mathbf{B}_2|\mathbf{B}_1^{\complement}\right)\cdot R\left(\mathbf{B}_1^{\complement}\right)=\frac{\int_X \boldsymbol{\mu}_{\{A_{0010}\}}(x)\cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{0010},A_{0011},A_{0001}\}}(x)\cdot f(x)\,dx}\cdot 0.9,\\
b &= R\left(\mathbf{B}_3|\mathbf{B}_1^{\complement}\right)\cdot R\left(\mathbf{B}_1^{\complement}\right)=\frac{\int_X \boldsymbol{\mu}_{\{A_{0011},A_{0001}\}}(x)\cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{0010},A_{0011},A_{0001}\}}(x)\cdot f(x)\,dx}\cdot 0.9,\\
c &= R\left(\mathbf{B}_1|\mathbf{B}_3^{\complement}\right)\cdot R\left(\mathbf{B}_3^{\complement}\right)=\frac{\int_X \boldsymbol{\mu}_{\{A_{1000},A_{0100},A_{0110}\}}(x)\cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{1000},A_{0100},A_{0110},A_{0010}\}}(x)\cdot f(x)\,dx}\cdot 0.5,\\
d &= R\left(\mathbf{B}_2|\mathbf{B}_3^{\complement}\right)\cdot R\left(\mathbf{B}_3^{\complement}\right)=\frac{\int_X \boldsymbol{\mu}_{\{A_{0010}\}}(x)\cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{1000},A_{0100},A_{0110},A_{0010}\}}(x)\cdot f(x)\,dx}\cdot 0.5.
\end{aligned}$$

So, we know matrix $\overleftrightarrow{R(\mathbf{B}|\mathbf{A})}$ and we can compute posteriori probabilities of floppy sets $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$. But sometimes we need to know the probability of some other floppy set. For example, what is the probability that the water is unhealthily cold?

We use again a priori probabilities and law of total probability:

$$\begin{aligned}
R^P(\{A_{1000}\}) &= R(\{A_{1000}\}|\mathbf{B}_1)\cdot R^P(\mathbf{B}_1)=\\
&= \frac{\int_X \boldsymbol{\mu}_{\{A_{1000}\}}(x)\cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\{A_{1000},A_{0100},A_{0110}\}}(x)\cdot f(x)\,dx}\cdot R^P(\mathbf{B}_1).
\end{aligned}$$

6.3 Calculation of the mean value without knowledge of the probability distribution

Sometimes the probability distribution is hardly determinable. This is true in particular for subjective variables, such as “need for something”. In this case, instead of the probability distribution, we can estimate mean values of individual floppy sets and calculate the mean from them.

The mean value of floppy set $\mathbf{B}_j$ can be computed for discrete case this way:

$$\begin{aligned}
\langle\mathbf{B}_j\rangle = E(x_i|\mathbf{B}_j) &= \sum_{x_i\in X} x_i\cdot R(x_i|\mathbf{B}_j)=\\
&= \frac{1}{R(\mathbf{B}_j)}\sum_{x_i\in X} x_i\cdot R(\mathbf{B}_j|x_i)\cdot R(x_i)=\\
&= \frac{\sum_{x_i\in X} x_i\cdot \boldsymbol{\mu}_{\mathbf{B}_j}(x_i)\cdot P(x_i)}{\sum_{x_i\in X} \boldsymbol{\mu}_{\mathbf{B}_j}(x_i)\cdot P(x_i)}
\end{aligned}$$

and for continuous case this way:

$$\begin{aligned}
\langle\mathbf{B}_j\rangle = E(x|\mathbf{B}_j) &= \int_X x\cdot f(x|\mathbf{B}_j)\,dx=\\
&= \frac{1}{R(\mathbf{B}_j)}\int_X x\cdot R(\mathbf{B}_j|x)\cdot f(x)\,dx=\\
&= \frac{\int_X x\cdot \boldsymbol{\mu}_{\mathbf{B}_j}(x)\cdot f(x)\,dx}{\int_X \boldsymbol{\mu}_{\mathbf{B}_j}(x)\cdot f(x)\,dx}.
\end{aligned}$$

If the properties on page 488 are satisfied, the mean value of x or x_i can be computed this way:

$$E\left(x\right)=E\left(\langle\mathbf{B}_j\rangle\right)=\sum_j\langle\mathbf{B}_j\rangle\cdot R^P\left(\mathbf{B}_j\right).$$

Example: A system of traffic lights is described by these rules:

1. If there is green signal on the traffic light and the time of green signal is very long, then the "need for green" is very low.

2. If there is red signal on the traffic light and the time of red signal is very long, then the "need for green" is very high.

3. ...

The output variable "need of green" is described by several fuzzy sets: A_0 – zero, A_1 – very low, A_2 – low, A_3 – normal, A_4 – high, A_5 – very high.

We know the probabilities of floppy sets $\mathbf{B}_j=\{A_j\}$. These probabilities are written in column vector $\overrightarrow{R^P(\mathbf{B})}$.

We do not know the probability distribution of "need for green", so we must estimate the mean values of those floppy sets. We write them into a row vector. For example this way:

$$\overrightarrow{\langle\mathbf{B}\rangle}=\left(0,\frac{1}{10},1,3,5,10\right).$$

We calculate the mean value of "need for green":

$$E\left(x\right)=\overrightarrow{\langle\mathbf{B}\rangle}\cdot\overrightarrow{R^P(\mathbf{B})}.$$

We see that if we sometimes describe some alternatives by numeric values, we can understand these numerical values as mean values of the relevant floppy sets.

7. Conclusion

We tried to show that floppy logic is a very useful, effective and relatively simple tool for solution of wide range of problems. I would be very pleased if it succeeded.

In this article, we solved problems of the following type: fuzzification – application of rules of system – defuzzification. There are, of course, many other areas where floppy logic could be useful. For example, estimation of parameters (e.g. [3, 8, 16]) or clusters searching (e.g. [1, 2, 7, 12, 13]). I believe that a reader who understood this and the previous article [14] would already be able to apply the floppy logic in new areas by himself.

Curriculum Vitae

Pavel Provinský, born in Prague, graduated from Charles University in Prague, Faculty of Mathematics and Physics in 2005. During my studies, I taught at my alma mater, then at the Private Nature Grammar School and at the University of Economics in Prague. I am a happy husband and father of three children. I am currently finalizing my PhD studies and at the same time I am teaching in the position of assistant professor at the Czech Technical University in Prague, Faculty of Transportation Sciences. The floppy logic is my PhD subject. I have already written one article on this topic [14], in which I explained the theoretical foundations. In the next article I intend to compare fuzzy and floppy logic.

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