CONSTRUCTION OF THE POLYGONAL FUZZY NEURAL NETWORK AND ITS APPROXIMATION BASED ON K-INTEGRAL NORM

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Abstract: The concept of an n-equidistant polygonal fuzzy number is introduced to avoid the complexity of the operations between fuzzy numbers. Firstly, the properties of linear operations and the convergence of n-equidistant polygonal fuzzy numbers are discussed, the method how to change a fuzzy number into an n-equidistant polygonal fuzzy number is shown. Next, for given a $\hat{\mu}$-integrable polygonal fuzzy valued function, an n-equidistant polygonal fuzzy valued function is constructed. By introducing the definition of K-quasi-additive integral and K-integral norm, the universal approximation of polygonal fuzzy neural network are studied. The final result indicates that the polygonal fuzzy neural network still possess universal approximation to an integrable system.

Key words: *Polygonal fuzzy numbers, quasi-additive integral, integrable polygonal fuzzy valued functions, integral norm, polygonal fuzzy neural network, universal approximation*

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1. Introduction

A fuzzy neural network (FNN) is an organic combination of an artificial neural network and fuzzy techniques, that form a hybrid intelligent system with both intelligent information processing and adaptability. In real life, it can effectively handle natural language messages, and there are more data messages of digital type than language messages. Thus, we can obtain date messages with corresponding input-output relationship of a fuzzy system by measurement date and transmission. Buckley [2–3] gave a conjecture when he studied the universal approximation of the regular FNN in 1994, and predicted the regular FNN is a universal approximation of the continuously increasing fuzzy function class. Since then, the network is

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systematically studied by many scholars at home and abroad on the system approximation and the learning algorithm. Consequently, some useful achievements [1, 4–6, 8, 11–15, 25] were acquired. All of these have important value for fuzzy inference, fuzzy control and image restoration technique. Liu [12] defined n-symmetrical polygonal fuzzy numbers for the first time in 2002, and completeness and separability of this space were discussed in detail. Furthermore, the universal approximation of a polygonal fuzzy neural network (PFNN) was researched. Unfortunately Liu [11–12] did not use integral norms to investigate them. In fact, a PFNN is established in accordance with the polygonal fuzzy numbers finish a fuzzy information processing by means of finite points on x-axis, and a PFNN directly effected relative operations on corresponding fuzzy numbers. In [12], it was proved that a three-layer feed-forward PFNN could be regarded as the universal approximation of the continuous increasing fuzzy functions. In the real world, most systems don't certainly satisfy continuity. Hence, it is necessary to generalize the continuity of fuzzy functions.

This paper is organized as follows: The method of transforming an ordinary fuzzy numbers into a polygonal fuzzy number is indicated in Section 2, and some basic concepts and the properties of the polygonal fuzzy number are briefly summarized. In Section 3, the concept of K-integral norm is given by introducing K-quasi-additive integral. In addition, a concrete PFNN is constructed in Section 4. In Section 5, approximation of the three-layer PFNN in the sense the K-integral norm is proved.

2. Equidistant polygonal fuzzy numbers

Although, some of the system theory based on fuzzy number had successfully applied in fuzzy control field. However, these operations of fuzzy arithmetic are extremely complex, even then the operations for the simple trigonometric fuzzy numbers and ladder fuzzy numbers are very difficult. The reason is that the four arithmetic operations in Zadeh's extension principle do not satisfy closeness. Thus, how to approximately finish nonlinear operations for general fuzzy numbers is an important problems.

Let $\mathbb{R}^+ = [0, +\infty), \mathbb{R}^d$ denote a d-dimensional Euclidean space, $\mathbb{N}$ denote natural number set, $\| \cdot \|$ be a norm in $\mathbb{R}^d$. For arbitrary $A, B \subset \mathbb{R}^d$, define

$$d_{\mathrm{H}}(A, B) = \max\{\bigvee_{x\in A}\bigwedge_{y\in B} \| x - y \|, \bigvee_{y\in B}\bigwedge_{x\in A} \| x - y \|\}.$$

By [7], we know that $d_{\mathrm{H}}(A, B)$ is an Hausdorff distance between A and B. Especially, $d_{\mathrm{H}}(A, B) =| a - c | \vee | b - d |$ whenever $A = [a, b], B = [c, d] \subset \mathrm{R}$.

Definition 2.1 *Let* $\widetilde{A} : \mathbb{R} \longrightarrow [0, 1]$*. If* $\widetilde{A}$ *satisfies the conditions (1)-(2):*

(1) $\ker(\widetilde{A}) = \{x \in \mathbb{R} \mid \widetilde{A}(x) = 1\} \neq \emptyset$*;*

(2) for any $\lambda \in (0, 1]$*, the* $\widetilde{A}_\lambda = \{x \in \mathbb{R} \mid \widetilde{A}(x) \geq \lambda\}$ *is a bounded closed interval. Then* $\widetilde{A}$ *is called a fuzzy number on* $\mathbb{R}$.

Let $F_0(\mathbb{R})$ denote the family of all fuzzy numbers on $\mathbb{R}$. Form [7], for any $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, we define

$$D(\widetilde{A}, \widetilde{B}) = \bigvee_{\lambda \in [0,1]} d_{\mathrm{H}}(\widetilde{A}_\lambda, \widetilde{B}_\lambda).$$

Then it shows that $(F_0(\mathbb{R}), D)$ constitutes a completely metric space.

The definition of the polygonal fuzzy numbers is put forward first time in [12], it has the excellent linear properties, this makes the operations of fuzzy numbers become quite succinct. In this section, we shall first introduce a kind of particular polygonal fuzzy numbers, please refer to Definition 2.2b.

Definition 2.2a (Liu [12]) *Let $\widetilde{A} \in F_0(\mathbb{R})$, for a given $n \in \mathbb{N}$. If there exists a group of ordered real numbers $a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2 \in \mathbb{R}$ with $a_0^1 \le a_1^1 \le \cdots \le a_n^1 \le a_n^2 \le \cdots \le a_1^2 \le a_0^2$ such that $\widetilde{A}(x)$ takes straight lines in interval $[a_{i-1}^1, a_i^1]$ and $[a_i^2, a_{i-1}^2]$, $i = 1, 2, \ldots, n$, i.e., for all $x \in \mathbb{R}$,*

$$\widetilde{A}(x) = \begin{cases} \widetilde{A}(a_{i-1}^1) + \frac{(x - a_{i-1}^1)(\widetilde{A}(a_i^1) - \widetilde{A}(a_{i-1}^1))}{(a_i^1 - a_{i-1}^1)}, & x \in [a_{i-1}^1, a_i^1], i = 1, 2, \ldots, n \\ 1, & x \in [a_n^1, a_n^2] \\ \widetilde{A}(a_{i-1}^2) + \frac{(a_{i-1}^2 - x)(\widetilde{A}(a_i^2) - \widetilde{A}(a_{i-1}^2))}{(a_{i-1}^2 - a_i^2)}, & x \in [a_i^2, a_{i-1}^2], i = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

Then $\widetilde{A}$ is called a n-polygonal fuzzy number, it is written as $\widetilde{A} = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2)$, where $(\cdot)$ is not a vector, it is only an expression of $\widetilde{A}$, and define $\frac{0}{0} = 0$. Please refer to Fig. 1.

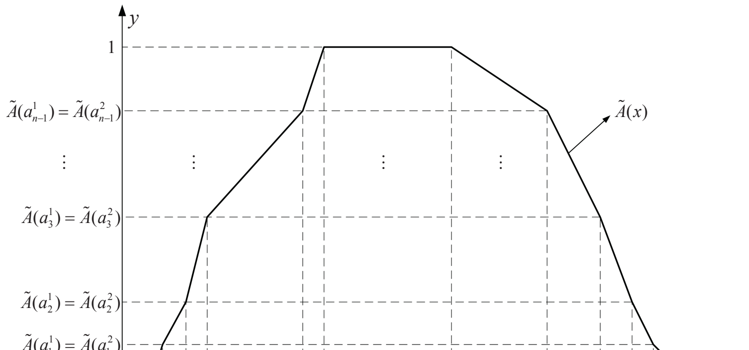

Fig. 1 *An n-polygonal fuzzy number $\widetilde{A}$.*

Obviously, 1-polygonal fuzzy number $\widetilde{A}$ degenerates a ladder fuzzy number or a trigonometric fuzzy number whenever $n = 1$. In addition, it is clearly that

$$\widetilde{A}(a_0^1) < \widetilde{A}(a_1^1) < \cdots < \widetilde{A}(a_n^1) = 1 = \widetilde{A}(a_n^2) > \cdots > \widetilde{A}(a_1^2) > \widetilde{A}(a_0^2).$$

Definition 2.2b *Particularly, in the sense of the partition in Definition 2.2a, we divide a closed interval* $[0,1]$ *of y-axis into equal n small intervals for a given* $n \in \mathbb{N}$. *Then* $\widetilde{A}(a_i^q) = \frac{i}{n}$ *and* $\widetilde{A}(a_i^q) - \widetilde{A}(a_{i-1}^q) = \frac{1}{n}$, $q = 1, 2;\ i = 1, 2, \ldots, n$, *and its membership function is expressed as follows:*

$$\widetilde{A}(x) = \begin{cases} \frac{i-1}{n} + \frac{(x - a_{i-1}^1)}{n(a_i^1 - a_{i-1}^1)}, & x \in [a_{i-1}^1, a_i^1], i = 1, 2, \ldots, n \\ 1, & x \in [a_n^1, a_n^2] \\ \frac{i-1}{n} + \frac{(a_{i-1}^2 - x)}{n(a_{i-1}^2 - a_i^2)}, & x \in [a_i^2, a_{i-1}^2], i = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

Then $\widetilde{A}$ *is called an n-equidistant polygonal fuzzy number, refer to Fig. 2.*

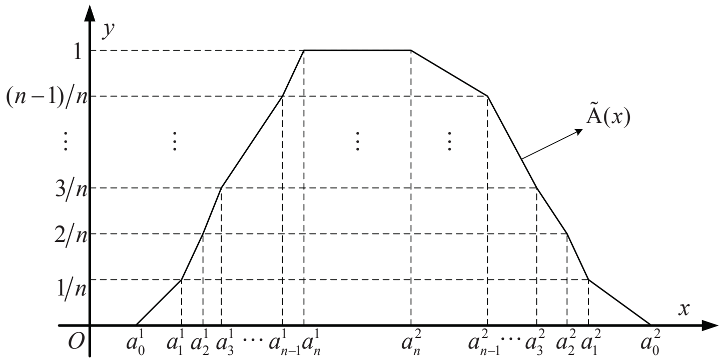

Fig. 2 *An n-equidistant polygonal fuzzy number* $\widetilde{A}$.

For a given $n \in \mathbb{N}$, let the symbol $Z_n(F_0(\mathbb{R}))$ denote the family of all n-equidistant polygonal fuzzy numbers on $F_0(\mathbb{R})$.

Note 1 For simplicity, in this paper, we will always discuss the problems in the sense of n-equidistant polygonal fuzzy numbers. From Definition 2.2b, some properties of n-equidistant polygonal fuzzy numbers are similar as a ladder or trigonometric fuzzy number. For given $n \in \mathbb{N}$, an n-equidistant polygonal fuzzy number $\widetilde{A}$ can be solely determined by $2n+2$ real numbers $a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2$ on $\mathbb{R}$. On the contrary, the $2n+2$ real numbers of satisfied above conditions can solely determined the analytic expression of an n-equidistant polygonal fuzzy number.

Example 2.1 *Let* $\widetilde{A} = (-2, -1, -0.5, 0.8, 1, 3) \in Z_n(F_0(\mathbb{R}))$. *The analytic expression of the n-equidistant polygonal fuzzy number can solely determined.*

In fact, let $2n+2 = 6$, it implies $n = 2$, then there exist a sole divided point $\lambda = \frac{1}{2}$. At the moment, we obtain the coordinates of the knots points on the image of a 2-equidistant polygonal fuzzy number, they are

$$(-2, 0), (-1, 0.5), (-0.5, 1), (0.8, 1), (1, 0.5), (3, 0).$$

Connecting neighbor knots points in order with straight line segments, we can get the analytic expression of 2-equidistant polygonal fuzzy number $\widetilde{A}(x)$, i.e.,

$$\widetilde{A}(x)=\begin{cases}\frac{1}{2}x+1, & -2\le x<-1\\ x+\frac{3}{2}, & -1\le x<-0.5\\ 1, & -0.5\le x\le 0.8\\ -\frac{5}{2}x+3, & 0.8<x\le 1\\ -\frac{1}{4}x+\frac{3}{4}, & 1<x\le 3\end{cases}$$

On the image of the corresponding $\widetilde{A}(x)$, please refer to Fig. 3.

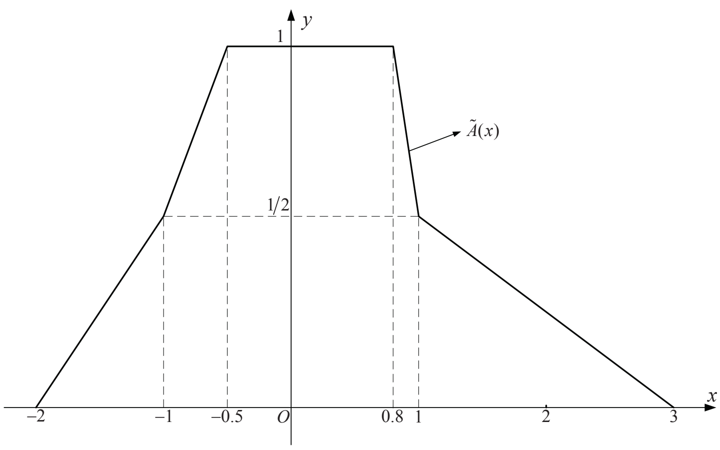

Fig. 3 *2-polygonal fuzzy number* $\widetilde{A}=(-2,-1,-0.5,0.8,1,3)$.

Clearly, $Z_n(F_0(\mathbb{R}))\subset F_0(\mathbb{R})$. As for a given fuzzy number, its corresponding n-equidistant polygonal fuzzy number depends on the selection of n, the bigger valued of n takes, the more knots of the polygonal lines are, the approximation capability of polygonal fuzzy numbers to given fuzzy numbers is more stronger, at the moment, they are becoming more and more complex.

Definition 2.3 (Wang [21]) *Let* $\widetilde{A},\widetilde{B}\in F_0(\mathbb{R}), Z_n(\widetilde{A})=(a_0^1,a_1^1,\ldots,a_n^1,a_n^2,\ldots,a_1^2,a_0^2), Z_n(\widetilde{B})=(b_0^1,b_1^1,\ldots,b_n^1,b_n^2,\ldots,b_1^2,b_0^2)\in Z_n(F_0(\mathbb{R}))$. *For a given* $n\in\mathbb{N}$, *define addition, subtraction and multiplication etc. as follows:*

(1) $Z_n(\widetilde{A})+Z_n(\widetilde{B})=(a_0^1+b_0^1,a_1^1+b_1^1,\ldots,\ a_n^1+b_n^1,a_n^2+b_n^2,\ldots,\ a_1^2+b_1^2,a_0^2+b_0^2)$;

(2) $Z_n(\widetilde{A})-Z_n(\widetilde{B})=(a_0^1-b_0^2,a_1^1-b_1^2,\ldots,\ a_n^1-b_n^2,a_n^2-b_n^1,\ldots,\ a_1^2-b_1^1,a_0^2-b_0^1)$;

(3) $Z_n(\widetilde{A})\cdot Z_n(\widetilde{B})=(c_0^1,c_1^1,\ldots,c_n^1,c_n^2,\ldots,c_1^2,c_0^2)$; $c_i^1=a_i^1b_i^1\wedge a_i^1b_i^2\wedge a_i^2b_i^1\wedge a_i^2b_i^2$; $c_i^2=a_i^1b_i^1\vee a_i^1b_i^2\vee a_i^2b_i^1\vee a_i^2b_i^2,\ i=0,1,2,\ldots,n$;

(4) $k\cdot Z_n(\widetilde{A})=(ka_0^1,ka_1^1,\ldots,ka_n^1,ka_n^2,\ldots,ka_1^2,ka_0^2), k\ge 0$.

Theorem 2.1 (Wang [21]) *Let* $\widetilde{A},\widetilde{B}\in F_0(\mathbb{R}),\forall n\in\mathbb{N}$. *Then the following properties hold as follows:*

(1) $Z_n(\widetilde{A}\pm\widetilde{B})=Z_n(\widetilde{A})\pm Z_n(\widetilde{B}),\quad Z_n(\widetilde{A}\cdot\widetilde{B})=Z_n(\widetilde{A})\cdot Z_n(\widetilde{B})$;

(2) $Z_n(Z_n(\widetilde{A}))=Z_n(\widetilde{A}),\quad Z_n(k\cdot\widetilde{A})=k\cdot Z_n(\widetilde{A})$, where $k\ge 0$.

Example 2.2 *Let* $\widetilde{A} = (-2, -1, -0.5, 0.8, 1, 3), \widetilde{B} = (-1, -0.5, 0, 1, 1.5, 2) \in Z_2(F_0(\mathbb{R}))$. *By Definition 2.3 and Theorem 2.1, its arithmetic operations are obtained*

$$\widetilde{A} + \widetilde{B} = (-3, -1.5, -0.5, 1.8, 2.5, 5);$$
$$\widetilde{A} - \widetilde{B} = (-4, -2.5, -1.5, 0, 8, 1.5, 4);$$
$$\widetilde{A} \cdot \widetilde{B} = (-4, -1.5, -0.5, 0.8, 1.5, 6).$$

Evidently, the space $Z_n(F_0(\mathbb{R}))$ of n-equidistant polygonal fuzzy numbers is closed with respect to the arithmetic operations, its extension operations are more simpler than the corresponding operations in Zadeh's extension principle, which possess excellent properties.

Aim at the (1) of Definition 2.3, please refer to the demonstration of Fig. 4.

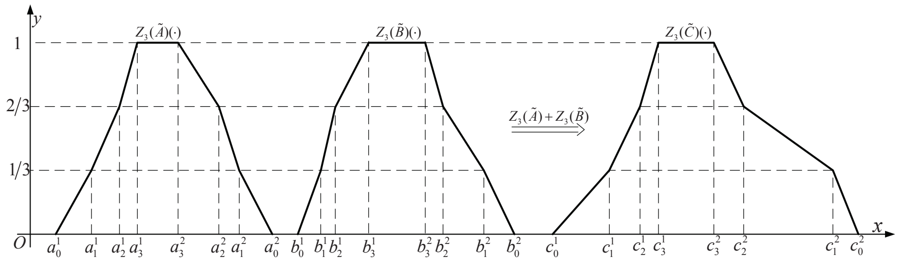

Fig. 4 *Addition of the polygonal fuzzy numbers* $\widetilde{A}$ *and* $\widetilde{B}$.

Theorem 2.2 (Wang [21]) *Let* $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R}), Z_n(\widetilde{A}) = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2), Z_n(\widetilde{B}) = (b_0^1, b_1^1, \ldots, b_n^1, b_n^2, \ldots, b_1^2, b_0^2) \in Z_n(F_0(\mathbb{R}))$. *Then we have*

$$D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) = \bigvee_{i=0}^{n} (\mid a_i^1 - b_i^1 \mid \vee \mid a_i^2 - b_i^2 \mid).$$

Definition 2.4 *Let a function* $\sigma : \mathbb{R} \to \mathbb{R}$. *If* σ *is bounded with* $\lim_{x\to-\infty} \sigma(x) = 0$, $\lim_{x\to+\infty} \sigma(x) = 1$. *Then* σ *is called an activation function on* $\mathbb{R}$. For example, the Sigmodial function $\sigma(x) = \frac{1}{1+e^{-x}}$ is an ordinary activation function.

Note 2 For given $n \in \mathbb{N}$, let an activation function σ is monotone increasing. Then σ may be extended as follows $\sigma : Z_n(F_0(\mathbb{R})) \to Z_n(F_0(\mathbb{R}))$. If $\widetilde{A} = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2)$. Then we have

$$\sigma(\widetilde{A}) = (\sigma(a_0^1), \sigma(a_1^1), \ldots, \sigma(a_n^1), \sigma(a_n^2), \ldots, \sigma(a_1^2), \sigma(a_0^2)).$$

Theorem 2.3 (Wang [21]) *Let* $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, *a given* $n \in \mathbb{N}$. *Then* $D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) \leq D(\widetilde{A}, \widetilde{B})$ *and* $\lim_{n\to-\infty} D(\widetilde{A}, Z_n(\widetilde{A})) = 0$.

Theorem 2.4 *Let* $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$. *Then* $\lim_{n\to-\infty} D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) = D(\widetilde{A}, \widetilde{B})$.

Proof. By Theorem 2.3, it is straightforward to see that

$$\lim_{n\to\infty} D(\widetilde{A}, Z_n(\widetilde{A})) = 0, \quad \lim_{n\to\infty} D(\widetilde{B}, Z_n(\widetilde{B})) = 0.$$

Therefore, by use of the definition of the limit of the sequence of numbers, for any $\varepsilon > 0$, it shows that there exists natural numbers $N_1, N_2 \in \mathbb{N}$ such that

$$D(\widetilde{A}, Z_n(\widetilde{A})) < \frac{\varepsilon}{2}, \quad D(\widetilde{B}, Z_n(\widetilde{B})) < \frac{\varepsilon}{2}.$$

whenever $n > N_1, n > N_2$, respectively. Taking N =max(N_1, N_2), whenever $n > N$, we can derive from that

$$D(\widetilde{A}, \widetilde{B}) \leq D(\widetilde{A}, Z_n(\widetilde{A})) + D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) + D(Z_n(\widetilde{B}), \widetilde{B}).$$

Hence, we obtain

$$D(\widetilde{A}, \widetilde{B}) - D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

According to Theorem 2.3, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$, whenever $n > N$, it is not hard to see that

$$\mid D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) - D(\widetilde{A}, \widetilde{B}) \mid = D(\widetilde{A}, \widetilde{B}) - D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) < \varepsilon.$$

Consequently, we can get that $\lim_{n \to -\infty} D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) = D(\widetilde{A}, \widetilde{B})$.

Theorem 2.5 (Wang [21]) *For a given $n \in \mathbb{N}$. Then $(Z_n(F_0(\mathbb{R})), D)$ constitutes a completely separable metric space.*

Next, for an arbitrary $n \in \mathbb{N}$, we shall discuss how to solely determine an n-equidistant polygonal fuzzy number $Z_n(\widetilde{A})$ for a given fuzzy number $\widetilde{A} \in F_0(\mathbb{R})$. On the contrary, we must emphasize that its corresponding fuzzy numbers isn't sole for given an n-equidistant polygonal fuzzy number.

For a given $n \in \mathbb{N}$, let $Z_n : F_0(\mathbb{R}) \to Z_n(F_0(\mathbb{R}))$ be a mapping. Then $Z_n(\cdot)$ is said to be an n-equidistant polygonal operator. Concrete exchanging method as follows.

In fact, for arbitrary $\widetilde{A} \in F_0(\mathbb{R})$, divide unit closed interval $[0, 1]$ on y- axis into n equal parts, it means that insert $n-1$ dividing points $\lambda_i = \frac{i}{n}, i = 1, 2, \ldots, n-1$.

Since A_λ is a bounded closed interval for any $\lambda \in [0, 1]$, letting $\widetilde{A}(x) \geq \lambda_i = \frac{i}{n}, i = 1, 2, \ldots, n-1$ (the inequality is certainly to solve), then solve x with $a_i^1 \leq x \leq a_i^2$ in supp $\widetilde{A} \subset \mathbb{R}$, and satisfies

$$[a_n^1, a_n^2] \subset [a_{n-1}^1, a_{n-1}^2] \subset \cdots \subset [a_1^1, a_1^2] \subset [a_0^1, a_0^2].$$

Thus, we obtain a group of real numbers $a_i^q, i = 0, 1, 2, \ldots, n; q = 1, 2$ with

$$a_0^1 \leq a_1^1 \leq \cdots \leq a_n^1 \leq a_n^2 \leq \cdots \leq a_1^2 \leq a_0^2.$$

That is to say that $\widetilde{A}$ may be changed into a $n-$equidistant polygonal fuzzy number, denoted as $Z_n(\widetilde{A}) = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2)$.

On the other hand, let $\widetilde{A}_{\frac{i}{n}} = [a_i^1, a_i^2]$ for $\lambda_i = \frac{i}{n} \in [0, 1]$, where $i = 0, 1, 2, \ldots, n$. In turn, Connect the knot points

$$(a_0^1, 0), \left(a_1^1, \frac{1}{n}\right), \left(a_2^1, \frac{2}{n}\right), \ldots, (a_n^1, 1), (a_n^2, 1), \ldots, \left(a_2^2, \frac{2}{n}\right), \left(a_1^2, \frac{1}{n}\right), (a_0^2, 0)$$

which are the points on the curve of membership function $\widetilde{A}(x)$ with straight line segments in order. Thus, we can get one ladder polygonal with continuity from the right whenever $x < a_n^1$, and continuity from the left whenever $x > a_n^2$. Obviously, it is not hard to see that

$$\ker(Z_n(\widetilde{A})) = \ker\widetilde{A} = [a_n^1, a_n^2], \quad \mathrm{supp}(Z_n(\widetilde{A})) = \mathrm{supp}\widetilde{A} = [a_0^1, a_0^2];$$

$$(Z_n(\widetilde{A}))_{\frac{i}{n}} = \widetilde{A}_{\frac{i}{n}} = [a_i^1, a_i^2], \quad i = 0, 1, 2, \ldots, n.$$

Particularly, we don't distinguish between $\{a\}$ and a whenever $\widetilde{A}$ degenerates a single point set $\{a\}$ on $\mathbb{R}$, stipulating for $Z_n(\{a\}) = Z_n(a) = (a, a, \ldots, a, a, \ldots, a, a)$.

Example 2.3 *The two fuzzy numbers are given as follows:*

$$\widetilde{A}(x) = \begin{cases} \sqrt{4x+1} - 2, & \frac{3}{4} \le x < 2 \\ 1, & 2 \le x \le 3 \\ 3 - \sqrt{x+1}, & 3 < x \le 8 \\ 0, & \text{otherwise} \end{cases}, \quad \widetilde{B}(x) = \begin{cases} 4 - \frac{4}{x+1}, & 0 \le x < \frac{1}{3} \\ 1, & x = \frac{1}{3} \\ \frac{4}{x+1} - 2, & \frac{1}{3} < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $\mathrm{supp}\widetilde{A} = [\frac{3}{4}, 8]$, $\ker\widetilde{A} = [2, 3]$; $\mathrm{supp}\widetilde{B} = [0, 1]$, $\ker\widetilde{B} = [\frac{1}{3}, \frac{1}{3}]$.

In fact, whenever $n = 2$, as for fuzzy number $\widetilde{A}$, choose divided point $\lambda = \frac{1}{2}$, let $\widetilde{A}(x) = \sqrt{4x+1} - 2 = \frac{1}{2}$ whenever $x \in [\frac{3}{4}, 2)$, which implies $x = \frac{21}{16}$; Similarly, let $\widetilde{A}(x) = 3 - \sqrt{x+1} = \frac{1}{2}$ whenever $x \in [3, 8]$, which implies $x = \frac{21}{4}$. And so, we get

$$Z_2(\widetilde{A}) = \left(\frac{3}{4}, \frac{21}{16}, 2, 3, \frac{21}{4}, 8\right)$$

Whenever $n = 3$, select divided points $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$, let $\sqrt{4x+1} - 2 = \frac{1}{3}, \frac{2}{3}$ whenever $x \in [\frac{3}{4}, 2)$, respectively, we deduce that $x_1 = \frac{10}{9}, x_2 = \frac{55}{36}$; let $3 - \sqrt{x+1} = \frac{1}{3}, \frac{2}{3}$ whenever $x \in (3, 8]$, respectively, we infer that $x_1 = \frac{55}{9}, x_2 = \frac{40}{9}$.

Hence, we get a $3-$equidistant polygonal fuzzy number of $\widetilde{A}$, which is shown by Fig. 5.

$$Z_3(\widetilde{A}) = \left(\frac{3}{4}, \frac{10}{9}, \frac{55}{36}, 2, 3, \frac{40}{9}, \frac{55}{9}, 8\right).$$

Putting divided points $\lambda_1 = \frac{1}{4}, \lambda_2 = \frac{2}{4}, \lambda_3 = \frac{3}{4}$, then it is easily to obtain a 4-equidistant polygonal fuzzy number of $\widetilde{A}$, which is

$$Z_4(\widetilde{A}) = \left(\frac{3}{4}, \frac{65}{64}, \frac{21}{16}, \frac{105}{64}, 2, 3, \frac{65}{16}, \frac{21}{4}, \frac{105}{16}, 8\right).$$

Especially, aim at $n = 3$, obtain the following coordinates of the knots of $\widetilde{A}$ in order: $(\frac{3}{4}, 0), (\frac{10}{9}, \frac{1}{3}), (\frac{55}{36}, \frac{2}{3}), (2, 1), (3, 1), (\frac{40}{9}, \frac{2}{3}), (\frac{55}{9}, \frac{1}{3}), (8, 0)$.

Returning to $n = 3$, we are not hard to get the membership function of the 3-equidistant polygonal fuzzy number $Z_3(\widetilde{A})(x)$ of $\widetilde{A}$ as follow:

$$Z_3(\widetilde{A})(x) = \begin{cases} \frac{12}{13}x - \frac{9}{13}, & \frac{3}{4} \le x \le \frac{10}{9} \\ \frac{4}{5}x - \frac{5}{9}, & \frac{10}{9} < x \le \frac{55}{36} \\ \frac{17}{12}x - \frac{7}{17}, & \frac{55}{36} < x \le 2 \\ 1, & 2 < x \le 3 \\ -\frac{3}{13}x - \frac{22}{13}, & 3 < x \le \frac{40}{9} \\ -\frac{1}{5}x - \frac{14}{9}, & \frac{40}{9} < x \le \frac{55}{9} \\ -\frac{3}{17}x - \frac{24}{17}, & \frac{55}{9} < x \le 8 \end{cases} .$$

and the mixture figure of $\widetilde{A}(x)$ and $Z_3(\widetilde{A})(x)$ is shown in Fig. 5.

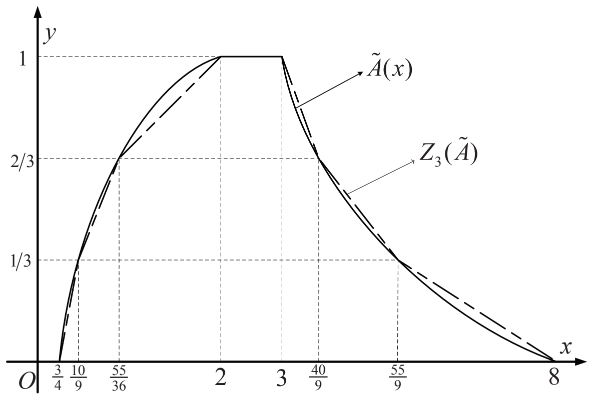

Fig. 5 *Mixture figure of* $\widetilde{A}$ *and* $Z_3(\widetilde{A})$.

By utilizing the similar method, as for fuzzy number $\widetilde{B}$, it is easily to show

$$Z_2(\widetilde{B}) = \left(0,\ \frac{1}{7},\ \frac{1}{3},\ \frac{1}{3},\ \frac{3}{5},\ 1\right);$$
$$Z_3(\widetilde{B}) = \left(0,\ \frac{1}{11},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{2},\ \frac{5}{7},\ 1\right);$$
$$Z_4(\widetilde{B}) = \left(0,\ \frac{1}{5},\ \frac{1}{8},\ \frac{3}{13},\ \frac{1}{3},\ \frac{1}{3},\ \frac{5}{11},\ \frac{3}{5},\ \frac{7}{9},\ 1\right).$$

In the light of Theorem 2.2, obviously, $D(Z_3(\widetilde{A}), Z_3(\widetilde{B})) = 7$. By Definition 2.3 and Theorem 2.1, we can clearly obtain

$$Z_3(\widetilde{A}+\widetilde{B}) = \left(\frac{3}{4}, \frac{119}{99}, \frac{311}{180}, \frac{7}{3}, \frac{10}{3}, \frac{89}{18}, \frac{430}{63}, 9\right);$$
$$Z_3(\widetilde{A}\cdot\widetilde{B}) = \left(0,\ \frac{10}{99},\ \frac{11}{36},\ \frac{2}{3},\ 1,\ \frac{20}{9},\ \frac{275}{63},\ 8\right).$$

Synthesizing the above discussion, we can see that n-equidistant polygonal fuzzy numbers make outstanding contributions for studying the operations and metric of fuzzy numbers by means of Theorem 2.1 to Theorem 2.5, it lays the theoretical foundation for further exploring equidistant polygonal fuzzy valued functions.

3. K-integral norms

In 1987, Sugeno [17] initially suggested the concepts of quasi-additive measures and quasi-additive integral. On the basis of these, the tK-integral, Kt-integral and a few new generalized fuzzy integrals were defined in [16, 19–20]. In 2011, Wang [21] put forward K-quasi-additive fuzzy valued integrals by introducing induced operator K, and then, the polygonal fuzzy neural network [9, 18, 21–22] was introduced in the sense of integral norm. In this section, we firstly introduce the definition of K-quasi-additive integral, and give the concept of K-integral norm.

Definition 3.1 (Wang [21]) *Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function with strictly monotone increasing, and it is derivable on $(0, +\infty)$, moreover, satisfies $K(0) = 0, K(1) = 1$. Then K is called to be an induced operator on $\mathbb{R}^+$.*

Obviously, its converse operator K^{-1} exists,too, and it is strictly increasing. For example, for any $x \in \mathbb{R}^+$, let $K(x) = \sqrt{x}$ or $K(x) = \log_2(1 + x)$, it is not difficult to see that the K are induced operators.

Definition 3.2 (Jiang [10]) *Let K be an induced operator on $\mathbb{R}^+$. For arbitrary $a, b \in \mathbb{R}^+$, define their K-quasi sum and K-quasi-product as follows $a \oplus b = K^{-1}(K(a) + K(b))$, $\quad a \otimes b = K^{-1}(K(a) \cdot K(b))$.*

Theorem 3.1 (Wang [21]) *Let $\oplus$ and $\otimes$ be K-quasi-sum and K-quasi product, respectively. For any $a, b \in \mathbb{R}^+$, then the following conclusions hold*

(1) $a + b \le a \oplus b$ and $a + b \le a \oplus b$ iff $K(a + b) \le K(a) + K(b)$;

(2) $K(a \oplus b) = K(a) + K(b)$, $\quad K(a \otimes b) = K(a) \cdot K(b)$;

(3) $K^{-1}(a + b) = K^{-1}(a) \oplus K^{-1}(b)$, $\quad K^{-1}(a \cdot b) = K^{-1}(a) \otimes K^{-1}(b)$.

Definition 3.3 (Sugeno [17]) *Let $(X, \Re)$ be a measurable space, K be an induced operator, $\hat{\mu} : \Re \to [0, +\infty]$. If the conditions (1)-(4) are fulfilled as follows:*

(1) $\hat{\mu}(\emptyset) = 0$;

(2) If $A, B \in \Re$, and $A \cap B = \emptyset$, then $\hat{\mu}(A \cup B) = \hat{\mu}(A) \oplus \hat{\mu}(B)$;

(3) If $A_n \subset \Re$, and $A_n \uparrow A$, then $\hat{\mu}(A_n) \uparrow \hat{\mu}(A)$;

(4) If $A_n \subset \Re$, and $A_n \downarrow A$, and there exists n_0 such that $\hat{\mu}(A_{n_0}) < +\infty$, then $\hat{\mu}(A_n) \downarrow \hat{\mu}(A)$. Then $\hat{\mu}$ is called a K-quasi-additive measure, and corresponding triple $(X, \Re, \hat{\mu})$ is called a space of K-quasi-additive measure.

Definition 3.4 (Wang [21]) *Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measure, K an induced operator, f an nonnegative measurable function, $A \in \Re, T = \{A_1, A_2, \ldots, A_n\}$ be a finite measurable partition on X. Put $\int_A^{(K)} f \mathrm{d}\hat{\mu} = \sup_T S_K(f, T, A)$, where $S_K(f, T, A) = \oplus \sum_{i=1}^{n} (\inf_{x \in A_i \cap A} f(x) \otimes \hat{\mu}(A_i \cap A))$. Then $\int_A^{(K)} f \mathrm{d}\hat{\mu}$ is called a K-quasi-additive integral of f with respect to $\hat{\mu}$ on A. Especially, f is called $\hat{\mu}$-integrable whenever integral value $\int_A^{(K)} f \mathrm{d}\hat{\mu} < +\infty$.*

Lemma 1(Jiang and Wang [10, 21]) *(Integral Transformation Theorem) Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measure, K be an induced operator, f be a nonnegative measurable function. Then $\int_A^{(K)} f \mathrm{d}\hat{\mu} = K^{-1}(\int_A K \circ f \mathrm{d}\hat{\mu})$ for each $A \in \Re$, where $\mu(\cdot) = K(\hat{\mu}(\cdot)), \mu$ is a Lebesgue measure.*

Note 3 In fact, a K-quasi-additive integral degenerates a Lebesgue integral whenever $K(x) = x$. Hence, this kind of integral is a generalization for Lebesgue integral, the corresponding quasi-sum and quasi-product degenerates ordinary sum and product, respectively. In addition, some excellent properties about them are very easily to be discovered, refer to [10, 19–22]. For simplicity, what we are going to discuss will be restricted on the space $Z_n(F_0(\mathbb{R}^+))$ in the following paper, here we define $\widetilde{A} \in Z_n(F_0(\mathbb{R}^+))$ iff $\widetilde{A}(x) = 0$ for arbitrary $x < 0$.

Definition 3.5 *Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measure $\hat{\mu}(E) < +\infty$. For any $E \in \Re, \widetilde{F} : E \to Z_n(F_0(\mathbb{R}^+))$ be an n-equidistant polygonal fuzzy valued function for a given $n \in \mathbb{N}$, or write it as $\widetilde{F}(x) = (f_0^1(x), f_1^1(x), \ldots, f_n^1(x), f_n^2(x), \ldots, f_1^2(x), f_0^2(x))$, for all $x \in E$. If each nonnegative real function $f_i^q : E \to \mathbb{R}^+ (i = 0, 1, 2, \ldots, n; q = 1, 2)$ are $\hat{\mu}$-integrable on E. Then F is said to be a $\hat{\mu}$-integrable polygonal fuzzy valued function on E, and stipulate for its integral*

$$\int_E \widetilde{F}(x)\mathrm{d}\mu = \left(\int_E f_0^1(x)\mathrm{d}\mu, \int_E f_1^1(x)\mathrm{d}\mu, \ldots, \int_E f_n^1(x)\mathrm{d}\mu, \right.$$
$$\left. \int_E f_n^2(x)\mathrm{d}\mu, \ldots, \int_E f_1^2(x)\mathrm{d}\mu, \int_E f_0^2(x)\mathrm{d}\mu \right).$$

Write $L^1(\hat{\mu}, Z_n) = \{\widetilde{F} : E \to Z_n(F_0(\mathbb{R}^+)) \mid \widetilde{F}$ is a $\hat{\mu}$-integrable polygonal fuzzy valued function on $E\}$. Moreover, from Definition 3.4, we can view that it is Lebesgue measurable if nonnegative real function $f_i^q(x)$ is $\hat{\mu}$-integrable.

Definition 3.6 *Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measure, K be an induced operator. For an arbitrary $n \in \mathbb{N}, \widetilde{F_1}, \widetilde{F_2} \in L^1(\hat{\mu}, Z_n), E \in \Re$, define $H(\widetilde{F_1}, \widetilde{F_2}) = \int_E^{(K)} D(\widetilde{F_1}(x), \widetilde{F_2}(x))\mathrm{d}\mu$. Then H is said to be a K-integral norm.*

According to Lemma 1 (Integral Transformation Theorem), we can obtain that $H(F_1, F_2)$ is also denoted as

$$H(\widetilde{F_1}, \widetilde{F_2}) = K^{-1}\left(\int_E K(D(\widetilde{F_1}(x), \widetilde{F_2}(x)))\mathrm{d}\mu\right).$$

Theorem 3.2 (Wang [21]) *Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measure, $n \in \mathbb{N}$. For an arbitrary $\widetilde{F_1}, \widetilde{F_2}, \widetilde{F_3} \in L^1(\hat{\mu}, Z_n)$, then $H(\widetilde{F_1}, \widetilde{F_3}) \leq H(\widetilde{F_1}, \widetilde{F_2}) \oplus H(\widetilde{F_2}, \widetilde{F_3})$.*

Theorem 3.3 (Wang [21]) *Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measure. For a given $n \in \mathbb{N}, (L^1(\hat{\mu}, Z_n), H)$ constitutes a metric space with respect to quasi-addition $\oplus$.*

4. Construction of a PFNN

Actually, the polygonal fuzzy neural network (PFNN) is a class of network system which connection weights and threshold value are taken valued in $Z_n(F_0(\mathbb{R}^+))$, and their inner operations are based on Definition 2.3, Theorem 2.1 and Note 2. In this section, we shall characterize the universal approximation of the three-layer regular PFNN in the sense of K-integral norm with respect to integrable fuzzy valued functions class.

Let a activation function $\sigma(\cdot)$ on knots be continuous, input neurons and output neurons be linear, input signal $x \in \mathbb{E} \subset \mathbb{R}$, connection weights $\tilde{V}_j, \tilde{U}_j \in Z_n(F_0(\mathbb{R}^+))$, threshold value $\tilde{\Theta} \in Z_n(F_0(\mathbb{R}^+))$. For any given $n \in \mathbb{N}$, we denote a three-layer regular PFNN as

$$Z_n(\widetilde{P_0}[\sigma]) = \{\widetilde{Y} : E \to Z_n(F_0(\mathbb{R}^+)) \mid \widetilde{Y}(x) = \sum_{i=1}^{p} \widetilde{V}_j \cdot \sigma(\widetilde{U}_j \cdot x + \widetilde{\Theta}_j),$$
$$p \in \mathbb{N}, \widetilde{V}_j, \widetilde{U}_j, \widetilde{\Theta}_j \in Z_n(F_0(\mathbb{R}^+)), \forall x \in \mathbb{E}\},$$

where p be the quantity of neurons in hidden layer, which is expressed by the following Fig. 6.

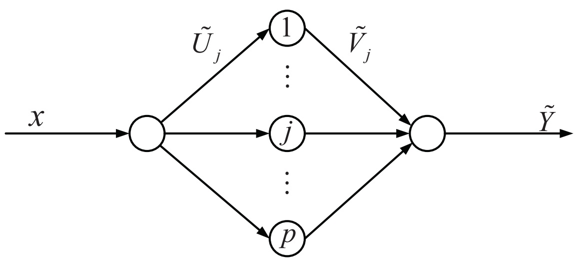

Fig. 6 *A single input single output PFNN.*

Definition 4.1 *For a given $n \in \mathbb{N}$, let $\Gamma = \{\widetilde{Y} : \mathbb{R} \to Z_n(F_0(\mathbb{R}))\}, \Omega \subset \Gamma$, $\forall \widetilde{F} \in \Gamma$. For an arbitrary compact set $U \subset \mathbb{R}$ and $\forall \varepsilon > 0$, there exists a $\widetilde{G} \in \Omega$ such that $D(\widetilde{F}(x), \widetilde{G}(x)) < \varepsilon$ for each $x \in U$. Then we call that Ω possess universal approximation with respect to $\widetilde{F}$, or say that Ω is an universal approximator of $\widetilde{F}$.*

Definition 4.2 (Liu [11]) *Let $\sigma : \mathbb{R} \to \mathbb{R}^+$ be a activation function, f be an arbitrary continuous function on $\mathbb{R}$. For an arbitrary $\varepsilon > 0$ and each compact set $U \subset \mathbb{R}$. If there exists p hidden neurons, connection weight $u_i, v_i \in \mathbb{R}$ and threshold value $\theta_i \in \mathbb{R}$ such that $\mid f(x) - \sum_{i=1}^{p} v_i \cdot \sigma(u_i \cdot x + \theta_i) \mid < \varepsilon$ for every $x \in U$. Then σ is called a Tauber-Wiener function.*

Next, we shall give a familiar theorem, this means that a continuous functions on a closed set $E \subset \mathbb{R}$ can be extended to the total straight line $\mathbb{R}$, and an extended theorem may be obtained.

Lemma 2 *Let $f(x), g(x)$ be continuous functions on $E \subset \mathbb{R}$. Then* $\min\{f(x), g(x)\}$ *is a continuous function on E, too.*

Proof. Since $\min\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) - \mid f(x) - g(x) \mid)$ for any $x \in E$. Obviously, the conclusion holds

Lemma 3 (Liu [12]) *Let a closed set $E \subset \mathbb{R}$, $f(x)$ be a continuous functions on E. Then there exists an extension function $g(x)$ such that it is continuous on $\mathbb{R}$, where $g(x) = f(x)$ for every $x \in E$.*

Lemma 4 (Liu [12]) *Let $\widetilde{F} : E \to Z_n(F_0(\mathbb{R}^+))$ be an n-equidistant polygonal fuzzy valued function, σ be a activation function. Then $\widetilde{F} \in Z_n(\widetilde{P_0}[\sigma])$ iff F may be* *denoted as* $\widetilde{F}(x) = (f_0^1(x), f_1^1(x), \ldots, f_n^1(x), f_n^2(x), \ldots, f_1^2(x), f_0^2(x))$ *for arbitrary* $x \in E$, *where* $f_j^q(x) = \sum_{i=1}^{p} h_j^q(i)(x)$, *and* $h_j^q(i)(x) = v_j(i,q) \cdot \sigma(w_j(i,q) \cdot x + \theta_j(i,q)), j = 0,1,2,\ldots,n; i = 1,2,\ldots,p; q = 1,2$, *and for each fixed* i, *the group of functions* $h_j^q(i)$ *satisfies*

$$h_j^1(i)(x) \le h_{j+1}^1(i)(x) \le h_{j+1}^2(i)(x) \le h_j^2(i)(x), j = 0,1,2,\ldots,n-1.$$

Theorem 4.1 *Let a closed set* $E \subset \mathbb{R}$, $f(x)$ *and* $g(x)$ *be continuous functions on* E *with* $f(x) \le g(x)$ *for each* $x \in E$. *Then there exists an extensive continuous function* $F(x)$ *and* $G(x)$ *of* $f(x)$ *and* $g(x)$ *on* $\mathbb{R}$, *respectively, such that* $F(x) \le G(x)$ *for arbitrary* $x \in \mathbb{R}$.

Proof. By Lemma 3, we know that the extensive continuous functions $F(x)$ and $G(x)$ exist. Next, we need only structure the continuous functions $F(x)$ and $G(x)$ with $F(x) \le G(x)$ for every $x \in \mathbb{R}$.

Virtually, selecting $F(x) = f(x), G(x) = g(x)$ whenever $x \in E$, and writing $U = \mathbb{R} - E$, then U is an open set on real straight line $\mathbb{R}$. Applying structure theorem of open sets, it follows that U can denoted as an union of finite or countable constitution intervals, i.e., $U = \cup_{k \in L}(a_k, b_k)$, where L is a countable index set.

(1) If all constitution intervals are finite, we connect point $(a_k, f(a_k))$ with $(b_k, f(b_k)), (a_k, g(a_k))$ with $(b_k, g(b_k))$ for any $k \in L$, respectively. Thus, for all $x \in (a_k, b_k)$, we get two equations of straight line on (a_k, b_k) as follows

$$\begin{aligned} F(x) &= f(a_k)\frac{b_k - x}{b_k - a_k} + f(b_k)\frac{x - a_k}{b_k - a_k}; \\ G(x) &= g(a_k)\frac{b_k - x}{b_k - a_k} + g(b_k)\frac{x - a_k}{b_k - a_k}, \forall x \in (a_k, b_k). \end{aligned}$$

On the other hand, in accordance with the definition of constitution intervals, it is easily to know that $a_k, b_k \notin U$, therefore, $a_k, b_k \in E$. By known conditions, it shows that $f(a_k) \le g(a_k), f(b_k) \le g(b_k)$. At the moment, we can infer that $F(x) \le G(x)$ for any $x \in (a_k, b_k)$.

(2) If there exists a constitution interval (a_{k_0}, b_{k_0}), which is an infinite interval, assume $(a_{k_0}, b_{k_0}) = (-\infty, b_{k_0})$ or $(a_{k_0}, +\infty)$, we define

$$F(x) = f(b_{k_0}), G(x) = g(b_{k_0}), \quad \forall x \in (-\infty, b_{k_0});$$

$$F(x) = f(a_{k_0}), G(x) = g(a_{k_0}), \quad \forall x \in (a_{k_0}, +\infty).$$

Clearly, $a_{k_0}, b_{k_0} \in E$ with $f(a_{k_0}) \le g(a_{k_0})$ and $f(b_{k_0}) \le g(b_{k_0})$. Hence, for every $x \in (a_{k_0}, b_{k_0})$, we derive from that $F(x) \le G(x)$, furthermore, it is straightforward to see $F(x) \le G(x)$ for any $x \in U$.

Consequently, for arbitrary $x \in \mathbb{R}, F(x) \le G(x)$ holds, too.

Theorem 4.2 *For any compact set $E \subset \mathbb{R}$ and a given $n \in \mathbb{N}, \widetilde{F} \in L^1(\hat{\mu}, Z_n)$, σ be a monotone increasing Tauber-Weiener function. Then there exists an $\widetilde{Q} \in Z_n(\widetilde{P_0}[\sigma])$ so that $\widetilde{Q}(x) = \sum_{i=1}^{p} \widetilde{V}_i \cdot \sigma(\widetilde{U}_i \cdot x + \widetilde{\Theta}_i)$ for arbitrary $x \in E$.*

Proof. Clearly, a set E is compact iff it is bounded closed set on Euclidean space. By hypothesis, for any $x \in E$, we let

$$\widetilde{F}(x) = (f_0^1(x), f_1^1(x), \ldots, f_n^1(x), f_n^2(x), \ldots, f_1^2(x), f_0^2(x)).$$

If appear certain two neighboring functions to be equal, then write them as one. Based on this consideration, without loss of generality, we suppose

$$0 \leq f_0^1(x) < f_1^1(x) < \cdots < f_n^1(x) < f_n^2(x) < \cdots < f_1^2(x) < f_0^2(x). \tag{1}$$

On account of $\widetilde{F} \in L^1(\hat{\mu}, Z_n)$, then F is $\hat{\mu}$-integrable on E, from Definition 3.5, it indicates that every nonnegative real functions $f_j^q(x)$ are Lebesgue integrable on E, where $f_j^q : E \to \mathbb{R}^+, j = 0, 1, 2, \ldots, n; q = 1, 2$.

Obviously, by Definition 3.5, the nonnegative real functions $f_j^q(x)$ are measurable. Now, with regard to each $f_j^q(x)$, making use of Lusin Theorem, there exists a closed subset $E_j^q \subset E$ for arbitrary $\delta > 0$ such that

$$\mu(E - E_j^q) < \frac{\delta}{2(n+1)},$$

where each functions $f_j^q(x)$ is continuous on E_j^q, $j = 0, 1, 2, \ldots, n; q = 1, 2$. As the closed set $E_j^q \subset E \subset \mathbb{R}$, in the light of Lemma 3, it follows that every continuous function $f_j^q(x)$ on E_j^q can be extended one continuous function $s_j^q(x)$ on E.

From another angle, we realize that (1) may be denoted as

$$f_j^1(x) < f_{j+1}^1(x) < f_{j+1}^2(x) < f_j^2(x), \quad j = 0, 1, 2, \ldots, n-1.$$

Utilizing Theorem 4.1, for all $x \in E$, we immediately obtain

$$s_j^1(x) \leq s_{j+1}^1(x) \leq s_{j+1}^2(x) \leq s_j^2(x), \quad j = 0, 1, 2, \ldots, n-1.$$

For arbitrary $x \in E$, let

$$\widetilde{S}(x) = (s_0^1(x), s_1^1(x), \ldots, s_n^1(x), s_n^2(x), \ldots, s_1^2(x), s_0^2(x)).$$

Evidently, $S(x)$ constitutes a continuous n-equidistant polygonal fuzzy valued function, where $s_j^q(x)$ is continuous on E whenever $j = 0, 1, 2, \ldots, n; q = 1, 2$.

On the other hand, with regard to each continuous function $s_j^q(x)$ on E, taking advantage of Definition 4.2, for any $\varepsilon > 0$, it shows that there exists a Tauber-Wiener function σ, natural numbers $p_{jq} \in \mathbb{N}$, connection weights $v'_{jq}(i), u'_{jq}(i) \in \mathbb{R}$ and threshold value $\theta'_{jq}(i) \in \mathbb{R}$ such that

$$\left| s_j^q(x) - \sum_{i=1}^{p_{jq}} v'_{jq}(i) \cdot \sigma(u'_{jq}(i) \cdot x + \theta'_{jq}(i)) \right| < \varepsilon, \tag{2}$$

where $j = 0, 1, 2, \ldots, n; q = 1, 2$, for any $x \in E$. Let $A = \bigcap_{j=0}^{n} E_j^1$, $B = \bigcap_{j=0}^{n} E_j^2$, then we can derive from

$$\mu(E - A) = \mu\left(\bigcup_{j=0}^{n}(E - E_j^1)\right) \leq \sum_{j=0}^{n}\mu(E - E_j^1) \leq \sum_{j=0}^{n}\frac{\delta}{2(n+1)} = \frac{\delta}{2}.$$

Similarly, it is not hard to see that $\mu(E - B) \leq \frac{\delta}{2}$. Furthermore, we obtain

$$\mu(E - A \cap B) \leq \mu(E - A) + \mu(E - B) < \delta.$$

Clearly, $A \cap B \neq \emptyset$. In fact, if $A \cap B = \emptyset$, adopt parallel method to above, we can get $\mu(E) = \mu(E - A \cap B) \leq \delta$. This contradicts that μ is a finite measure. At present, for arbitrary $x \in E$, let

$$h_j^q(x) = \sum_{i=1}^{p_{jq}} v'_{jq}(i) \cdot \sigma(u'_{jq}(i) \cdot x + \theta'_{jq}(i)), j = 0, 1, 2, \ldots, n; q = 1, 2. \tag{3}$$

Then (2) may be written as

$$\mid s_j^q(x) - h_j^q(x) \mid \leq \frac{\varepsilon}{2}. \tag{4}$$

For arbitrary $x \in E, j = 1, 2, \ldots, n$, let

$$\varphi_j(x) = \min\{s_j^1(x) - s_{j-1}^1(x), s_n^2(x) - s_n^1(x), s_{j-1}^2(x) - s_j^2(x)\}.$$

Since each real function $s_j^q(x)$ is continuous on E, from Lemma 2, we know that $\varphi_j(x)$ is continuous on E, too, and satisfy $\varphi_j(x) > 0, j = 1, 2, \ldots, n$. Putting

$$\delta_E(n) = \inf_{x \in E} \min_{1 \leq j \leq n} \varphi_j(x).$$

Virtually, for fixed $n \in \mathbb{N}$, denote $\psi_n(x) = \min_{1 \leq j \leq n} \varphi_j(x)$. In the light of Lemma 2, we can obtain that positive function $\psi_n(x)$ is still continuous on E. Then minimum value of the function $\psi_n(x)$ is acquired on closed set E. Therefore, $\delta_E(n)$ is existing, and $\delta_E(n) \geq 0$.

Next, we are going to verify $\delta_E(n) \neq 0$. On the contrary, if $\delta_E(n) = 0$, i.e., $\inf_{x \in E}\psi_n(x) = 0$, we notice that continuous function $\psi_n(x)$ must acquire minimum value on closed set E, i.e., inf = min, thus, there exists $x_0 \in E$ such that

$$0 = \psi_n(x_0) = \min_{1 \leq j \leq n} \varphi_j(x_0).$$

Consequently, there exists a $j_0 \in \{1, 2, \ldots, n\}$ such that $\varphi_{j_0}(x_0) = 0$. This contradicts that all $\varphi_j(x) > 0$ on E. Hence, $\delta_E(n) > 0$.

At the moment, for given $n \in \mathbb{N}$ and for all $x \in E, j = 1, 2, \ldots, n$. It is easily to see that

$$0 < \delta_E(n) \leq \varphi_j(x) \leq (s_j^1(x) - s_{j-1}^1(x)) \wedge (s_{j-1}^2(x) - s_j^2(x)).$$

On the other hand, whenever $q = 1$, for arbitrary $\varepsilon > 0$, and restricting $\varepsilon < \delta_E(n)$, by (4), we obtain

$$s_j^1(x) - \frac{\varepsilon}{2} < h_j^1(x) < s_j^1(x) + \frac{\varepsilon}{2}, \quad j = 0, 1, 2, \ldots, n.$$

Therefore, it follows that

$$h^1_{j+1}(x) - h^1_j(x) > s^1_{j+1} - s^1_j(x) - \varepsilon > s^1_{j+1} - s^1_j(x) - \delta_E(n) \geq 0.$$

Similarly, whenever $q = 2$, from (4), we can infer

$$h^2_j(x) - h^2_{j+1}(x) > s^2_j - s^2_{j+1}(x) - \varepsilon > s^2_j - s^2_{j+1}(x) - \delta_E(n) \geq 0.$$

Synthesizing above discussion, we immediately can summarize that

$$0 \leq h^1_0(x) < h^1_1(x) < \cdots < h^1_n(x) < h^2_n(x) < \cdots < h^2_1(x) < h^2_0(x).$$

For each $x \in E$, let

$$\widetilde{Q}(x) = (h^1_0(x), h^1_1(x), \ldots, h^1_n(x), h^2_n(x), \ldots, h^2_1(x), h^2_0(x)). \tag{5}$$

According to Lemma 4, it indicates the $\widetilde{Q} \in Z_n(F_0(\mathbb{R}^+))$. Next, we are going to substitute for expression (3), for arbitrary $x \in E$, let

$$h^q_j(i)(x) = v'_{jq}(i) \cdot \sigma(u'_{jq}(i) \cdot x + \theta'_{jq}(i)), \quad j = 0, 1, 2, \ldots, n;\ q = 1, 2.$$

In the light of (5), we know that their coefficients satisfy

$$v'_{j1} \leq v'_{(j+1)1} \leq v'_{(j+1)2} \leq v'_{j2};$$

$$u'_{j1} \leq u'_{(j+1)1} \leq u'_{(j+1)2} \leq u'_{j2};$$

$$\theta'_{j1} \leq \theta'_{(j+1)1} \leq \theta'_{(j+1)2} \leq \theta'_{j2}.$$

Combining (3) and (5), for arbitrary $x \in E$, we get

$$\begin{aligned}\widetilde{Q}(x) \quad = \quad & \left(\sum_{i=1}^{p_{01}} h^1_0(i)(x), \sum_{i=1}^{p_{11}} h^1_1(i)(x), \ldots, \sum_{i=1}^{p_{n1}} h^1_n(i)(x), \sum_{i=1}^{p_{n2}} h^2_n(i)(x), \ldots,\right. \\ & \left. \sum_{i=1}^{p_{12}} h^2_1(i)(x), \sum_{i=1}^{p_{02}} h^2_0(i)(x)\right).\end{aligned} \tag{6}$$

Based on the above investigation, we shall adjust the coefficient terms $v'_{jq}(i), w'_{jq}(i)$ and $\theta'_{jq}(i)$ in the group of functions $h^q_j(x)$ by constituting three transformations, let $p = \sum_{j=0}^{n} p_{jq}$, $\beta_k = \sum_{j=0}^{k-1} p_{jq}$, and choose $\beta_1 = 0$, $k = 2, 3, \ldots, n$; $q = 1, 2$. Moreover, let

$$v^q_j = \begin{cases} v'_{(j-\beta_k)q}, & \beta_k < j \leq \beta_{k+1}, \\ 0, \text{otherwise}, \end{cases} \quad \theta^q_j = \begin{cases} \theta'_{(j-\beta_k)q}, & \beta_k < j \leq \beta_{k+1}, \\ 0, \text{otherwise}, \end{cases}$$

$$u^q_j = \begin{cases} u'_{(j-\beta_k)q}, & \beta_k < j \leq \beta_{k+1}, \\ 0, \text{otherwise}. \end{cases}$$

In accordance with above transformations, for arbitrary $j \in \{0, 1, 2, \ldots, n\}$; $q = 1, 2$, for any $x \in E$, it isn't difficult to prove

$$\sum_{i=1}^{p} v_j^q(i) \cdot \sigma(u_j^q(i) \cdot x + \theta_j^q(i)) = \sum_{i=1}^{p_{jq}} v'_{jq}(i) \cdot \sigma(u'_{jq}(i) \cdot x + \theta'_{jq}(i)). \tag{7}$$

Now, for arbitrary $i \in \{1, 2, \ldots, p\}$, let

$$\begin{aligned}
\widetilde{V}_i &= (v_0^1(i), v_1^1(i), \ldots, v_n^1(i), v_n^2(i), \ldots, v_1^2(i), v_0^2(i));\\
\widetilde{U}_i &= (u_0^1(i), u_1^1(i), \ldots, u_n^1(i), u_n^2(i), \ldots, u_1^2(i), u_0^2(i));\\
\widetilde{\Theta}_i &= (\theta_0^1(i), \theta_1^1(i), \ldots, \theta_n^1(i), \theta_n^2(i), \ldots, \theta_1^2(i), \theta_0^2(i)).
\end{aligned}$$

It means that there exists connection weights $\widetilde{V}_i, \widetilde{U}_i \in Z_n(F_0(\mathbb{R}^+))$ and threshold value $\widetilde{\Theta}_i \in Z_n(F_0(\mathbb{R}^+))$, $i = 1, 2, \ldots, p$. Substitute expression (6) by (7), in a moment, applying Definition 2.3, for arbitrary $x \in E$, we immediately derive from

$$\begin{aligned}
\widetilde{Q}(x) = &\Bigg(\sum_{i=1}^{p} v_0^1 \cdot \sigma(u_0^1(i) \cdot x + \theta_0^1(i)), \sum_{i=1}^{p} v_1^1 \cdot \sigma(u_1^1(i) \cdot x + \theta_1^1(i)), \ldots,\\
&\sum_{i=1}^{p} v_n^1 \cdot \sigma(u_n^1(i) \cdot x + \theta_n^1(i)), \sum_{i=1}^{p} v_n^2 \cdot \sigma(u_n^2(i) \cdot x + \theta_n^2(i)), \ldots,\\
&\sum_{i=1}^{p} v_1^2 \cdot \sigma(u_1^2(i) \cdot x + \theta_1^2(i)), \sum_{i=1}^{p} v_0^2 \cdot \sigma(u_0^2(i) \cdot x + \theta_0^2(i))\Bigg)\\
= &\sum_{i=1}^{p} \widetilde{V}_i \cdot \sigma(\widetilde{U}_i \cdot x + \widetilde{\Theta}_i).
\end{aligned}$$

Hence, aim at a given $\widetilde{F} \in L^1(\hat{\mu}, Z_n)$, we may construct an n-equidistant polygonal fuzzy valued function $\widetilde{Q}$ so that $\widetilde{Q} \in Z_n(\widetilde{P}_0[\sigma])$.

5. Approximation based on K-integral norm

Next, we are going to obtain the important result which is Theorem 5.1 by utilizing Theorem 4.2. The conclusions indicate that a PFNN still possess the capability of universal approximation to an integrable system.

Theorem 5.1 *Let $(\mathbb{R}, \Re, \hat{\mu})$ be a finite K-quasi-additive measure space. For given $n \in \mathbb{N}$, for any compact set $E \subset \mathbb{R}, \widetilde{F} \in L^1(\hat{\mu}, Z_n)$, σ be a monotone increasing Tauber-Wiener function. Then there exists an n-equidistant polygonal fuzzy valued function $\widetilde{Q} \in Z_n(\widetilde{P}_0[\sigma])$ such that $Z_n(\widetilde{P}_0[\sigma])$ can approximate to $\widetilde{F}$ by arbitrary accuracy in the sense of K-integral norm H.*

Proof. Take advantage of Theorem 4.2, it is straightforward to know the existence of an n-equidistant polygonal fuzzy valued function $\widetilde{Q}(x)$, and $\widetilde{Q} \in Z_n(\widetilde{P}_0[\sigma])$. Without loss of generality, for any $x \in E$, which is denoted by

$$\widetilde{Q}(x) = (h_0^1(x), h_1^1(x), \ldots, h_n^1(x), h_n^2(x), \ldots, h_1^2(x), h_0^2(x)),$$

where $h_j^q(x) = \sum_{i=1}^{p} v_j^q(i) \cdot \sigma(u_j^q(i) \cdot x + \theta_j^q(i)), j = 0, 1, 2, \ldots, n; q = 1, 2$.

Now, we are going to prove that the n-equidistant polygonal fuzzy valued function Q approximates to F in K-integral norm H.

Actually, for every $\varepsilon > 0$, by Theorem 4.2, taking $\delta = \varepsilon > 0$, we can realize that there exists closed sets $A, B \subset E$ with $A \cap B \neq \emptyset$ such that $\mu(E - A \cap B) < \delta$.

Furthermore, according to Theorem 4.1, it explicates $\widetilde{F}(x) = \widetilde{S}(x)$ for any $x \in A \cap B$. On the one hand, for given $n \in \mathbb{N}$, applying Theorem 2.2 and combining (4), for arbitrary $x \in A \cap B$, we can view that the distance function $D(\widetilde{F}(x), \widetilde{Q}(x))$ on $A \cap$B is denoted as follow:

$$\begin{aligned} D(\widetilde{F}(x), \widetilde{Q}(x)) &= \vee_{i=0}^{n}(\mid s_i^1(x) - h_i^1(x) \mid \vee \mid s_i^2(x) - h_i^2(x) \mid) \\ &\leq \vee_{i=0}^{n}(\frac{\varepsilon}{2} \vee \frac{\varepsilon}{2}) = \frac{\varepsilon}{2}. \end{aligned}$$

On the other hand, since $(\mathbb{R}, \Re, \hat{\mu})$ be a finite K-quasi-additive measure space, and $\mu(\cdot) = K(\hat{\mu}(\cdot))$ is a Lebesgue measure. By means of monotone continuity of induced operator K, it reveals that compound distance function $K(D(\widetilde{F}(x), \widetilde{Q}(x)))$ is Lebesgue integrable on E.

In addition, making use of absolute continuity of Lebesgue integrals, we deduce

$$\int_{E - A \cap B} K((\widetilde{F}(x), \widetilde{Q}(x)))\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Consequently, utilizing Lemma 1 and the Equation (2) of Theorem 3.1, we can infer that

$$\begin{aligned} H(\widetilde{F}, \widetilde{Q}) = \ & K^{-1}\left(\int_E K(D(\widetilde{F}(x), \widetilde{Q}(x)))\mathrm{d}\mu\right) \\ &= K^{-1}\left(\int_{A \cap B} K(D(\widetilde{F}(x), \widetilde{Q}(x)))\mathrm{d}\mu + \int_{E - A \cap B} K(D(\widetilde{F}(x), \widetilde{Q}(x)))\mathrm{d}\mu\right) \\ &= K^{-1}\left(\int_{A \cap B} K(D(\widetilde{F}, \widetilde{Q}))\mathrm{d}\mu\right) \oplus K^{-1}\left(\int_{E - A \cap B} K(D(\widetilde{F}, \widetilde{Q}))\mathrm{d}\mu\right) \\ &\leq K^{-1}\left(\int_{A \cap B} K(\frac{\varepsilon}{2})\mathrm{d}\mu\right) \oplus K^{-1}\left(\frac{\varepsilon}{2}\right) \\ &= K^{-1}\left(K(\frac{\varepsilon}{2})\mu(A \cap B) + \frac{\varepsilon}{2}\right). \end{aligned}$$

In fact, for any $\varepsilon > 0$, as K is strictly monotone increasing, $K(\frac{\varepsilon}{2})$ is an infinitesimal quantity, and $\mu(A \cap B) \leq \mu(E) < +\infty$. Hence, the expression $K(\frac{\varepsilon}{2})\mu(A \cap B) + \frac{\varepsilon}{2}$ is an infinitesimal quantity, too. In addition, the induced operator K^{-1} is strictly monotone increasing, it follows that the expression $K^{-1}(K(\frac{\varepsilon}{2})\mu(A \cap B) + \frac{\varepsilon}{2})$ is still arbitrarily small. Thus, by Definition 4.1, it explicates that this three-layer PFNN $Z_n(\widetilde{P}_0[\sigma])$ can approximate to $\widetilde{F}$ by any accuracy in K-integral norm H.

6. Conclusions

So far, we have obtained a three-layer polygonal fuzzy neural network (PFNN) still possess universal approximation with respect to a class of $\hat{\mu}$-integrable polygonal fuzzy valued functions. Because most systems don't certainly satisfy continuity in the real world, this forces us to generalize the continuity of approximate functions. In fact, the concepts of an n-equidistant polygonal fuzzy number and K-integral norm being introduced play an important role in our discussion. It is worth emphasizing that the paper focuses on the universal approximation of a three-layer PFNN for a class of integrable functions in the sense of K-integral norm. The results of investigation indicate that the approximation capability of the PFNN to past continuous fuzzy system may be generalized to general integrable systems. In addition, these conclusions suit the case of negative polygonal fuzzy numbers. Since a number of fuzzy messages may be characterized by positive or negative fuzzy numbers in practice. Consequently, these results will provide the applications of the PFNN models and soft-computing technique with the necessary theoretical basis.

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